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The Distribution Free Newsboy Problem with Partial Information

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Abstract

We present a new ordering rule for the newsboy problem where besides mean and variance of demand, the probability that the demand assuming the value zero is also known. We derive a lowerbound for the expected profit over the set of distributions with given parameters and construct a distribution which achieves the bound. We apply our analysis to an M/G/1 queue with server vacations, which is the base model for many production-inventory systems.

Key words: *Newsboy problem, Inventory Control, M/G/1 queue, Server vacations*

1 Introduction

The newsboy problem is to decide the stock quantity of an item when there is a single purchasing opportunity before the start of the selling period and the demand for the item is random. The trade off is between the risk of overstocking, which results in disposal at a price below the unit purchasing cost and the risk of understocking, which results in loss of opportunity for making a profit. For a discussion on the newsboy problem, see Nahmias [4]. In this problem, however, the optimal stock that maximizes the expected revenue depends on the *complete* distribution of the demand variable and in most cases, information on the distribution of demand is very limited.

However, lower bounds on expected profit can be obtained as a function of the mean and variance of the demand variable only. One such bound is due to Scarf [5] which surprisingly did not receive much attention until its recent resurrection in the works of Gallego and Moon [3]. It is also possible to identify a (unique) distribution which achieves the bound and study the influence of parameters on the expected profits in the worst case scenario.

Extending the above studies further, we analyze the lower bound on objective value in the case when partial information pertaining to the distribution is made available. Here in particular we analyze the case when probability of zero demand is known. The relevance of such analysis can be articulated in many contexts which include inventory management in manufacturing systems.

Motivation for our study comes from inventory control framework of any production-inventory system where the newsboy model is commonly used for analysis. Further, in a single stage multi-class production-inventory system, a problem of interest is to find optimal production sequence and base stock levels for various classes of items in the presence of backorder and holding costs. The dynamics of such systems are, in general, modeled using a single server queue with server vacations. However, in most cases, it is only possible to compute moments of various quantities of interest, such as *short fall* (inventory level - base stock level). With the knowledge of these moments, one can directly apply the solution to the distribution free newsboy model to determine approximate base stock levels. For instance, see Federgruen and Katalan [2] for a similar procedure. It is interesting to note that, in the previous scenario, it is also possible to compute, in addition to moments, the steady state probability of *shortfall* taking the value zero. The details of computation of this probability are presented in the sequel with reference to M/G/1 queue with server vacations. Since our analysis provides a tighter bound on the objective value, one can arrive at a better estimate of base stocks in the cases discussed above compared to the base- stocks derived using just the knowledge of moments.

This paper is organized as follows. In Section 2, we describe the newsboy problem framework in detail. In Section 2.1, we provide a lower bound on the expected profits over the set of distributions having same mean, variance and which assign same mass at zero. We construct the worst case distribution in Theorem 2. Our construction procedure is based on very simple and intuitive arguments. Subsequently, we compute the optimal profits and corresponding stock order quantity. Section 2.3 provides a numerical example. Since in many multi-class single stage production- inventory systems discuss an M/G/1 queue model with server vacations and derive the expression for the steady state probability of queuelength assuming the value zero.

2 Problem Description and Analysis

For the sake of completeness, we give a formal description of the newsboy problem. We follow the notation used in [3] for our presentation.

The newsboy problem is to determine optimal stock of an item when there exists a single purchasing opportunity before selling can start. The demand for the item is random. An item remaining unsold at the end of the period has a salvage value less than the unit cost of purchase. Let

$c > 0$	: the unit cost
$p = c(1+m) > c$	: the unit selling price
$s = (1-d)c < c$	: the unit salvage value
μ	: the expected demand
σ	: the standard deviation of the demand
δ	: the probability that demand is zero
Q	: the order quantity

Note that the mark-up m(discount d) is a positive parameter that indicates the return(loss) per dollar on units sold(unsold). Let D denote the random demand. Assume that no information regarding the distribution of D, but for its mean μ, standard deviation σ and the probability δ that the demand is zero, is known. Let the class of distributions with the known parameters be denoted by $\mathcal{G}$. In what follows, $x^+ := max[x, 0]$.

For any given distribution of demand, $G \in \mathcal{G}$, and for any order quantity Q, expected profit can be written as

$$\pi^G(Q) \;:=\; pE[min(Q,D)] + sE[Q-D]^+ - cQ \tag{2.1}$$

where $min(Q,D)$ corresponds to number of units sold and $(Q-D)^+$ corresponds to number of units salvaged.

Observe that,

$$\begin{aligned} min(Q,D) &= D - (D-Q)^+ \\ (Q-D)^+ &= (Q-D) + (D-Q)^+ \end{aligned}$$

Thus, the expected profit in (2.1) can be rewritten as

$$\begin{aligned} \pi^G(Q) &= (p-s)\mu - (c-s)Q - (p-s)E[D-Q]^+ \\ \Rightarrow \pi^G(Q) &= c\{(m+d)\mu - dQ - (m+d)E[D-Q]^+\} \end{aligned} \tag{2.2}$$

Hence, maximizing the expected profit given by (2.2) is equivalent to minimizing

$$\Gamma^G(Q) := dQ + (m+d)E[D-Q]^+ \tag{2.3}$$

If the distribution of the demand is completely known, the optimal order quantity, Q_c^*, which maximizes (2.2) is given by:

$$Q_c^* \;=\; \min\{k : P(D \le k) \ge \frac{m}{m+d}\} \tag{2.4}$$

In the absence of complete knowledge of the distribution of D, it would be useful to derive a lower bound on the expected profit and analyze (2.2) when the bound is achieved by any distribution in $\mathcal{G}$. With this perspective, we address the problem of finding the expected profit for the worst possible distribution (which achieves the lower bound referred to above) in $\mathcal{G}$.

2.1 Analysis

In the following we derive an upperbound for the expected value of $(D-Q)^+$ and construct a distribution which lies in $\mathcal{G}$ for which the bound is tight.

Theorem 2.1 *Under any distribution $G \in \mathcal{G}$, the following holds for all $Q > 0$*

$$E[D-Q]^+ \;\le\; \frac{\delta Q + \sqrt{(1-\delta)(\sigma^2 + (Q-\mu)^2 - \delta Q^2)} + (\mu - Q)}{2} \tag{2.5}$$

Proof
First, note that

$$(D-Q)^+ \;=\; \frac{|D-Q| + (D-Q)}{2} \tag{2.6}$$

$$|D-Q| = \begin{cases} Q & \text{w.p. } \delta \\ Y & \text{w.p. } (1-\delta) \end{cases}$$

for some random variable Y.

It follows that,

$$Var(|D-Q|) \;\ge\; \delta(1-\delta)(E[Y]-Q)^2 \tag{2.7}$$

Further,

$$E(D-Q)^2 \;=\; \sigma^2 + (Q-\mu)^2 \tag{2.8}$$

$$E|D-Q| \;=\; \delta Q + (1-\delta)E[Y]$$

$$\Rightarrow E[Y] \;=\; \frac{E|D-Q| - \delta Q}{(1-\delta)} \tag{2.9}$$

Using (2.9) in (2.7), we get:

$$E[(D-Q)^2] - E^2|D-Q| \geq \delta(1-\delta)\left(\frac{E|D-Q| - \delta Q}{(1-\delta)} - Q\right)^2$$
$$\Rightarrow E|D-Q| \leq \delta Q + \sqrt{(1-\delta)(E[(D-Q)^2] - \delta Q^2)} \quad (2.10)$$

Use (2.8) in (2.10) to get

$$E|D-Q| \leq \delta Q + \sqrt{(1-\delta)(\sigma^2 + (Q-\mu)^2 - \delta Q^2)} \quad (2.11)$$

Substitution of (2.11) in (2.6) gives (2.5). Δ.

Now, we construct a distribution, which achieves the bound in (2.5) by way of the following theorem:

Theorem 2.2 *For every Q, there exists a distribution $G^* \in \mathcal{G}$with given parameters for which the bound in Theorem (2.1) is tight.*

Proof

To prove that the bound in (2.5) is tight we need to construct a distribution of D, such that

$$D = \begin{cases} 0 & \text{w.p. } \delta \\ Y' & \text{w.p. } (1-\delta) \end{cases}$$

for some random variable Y' and,

$$|D-Q| = \begin{cases} Q & \text{w.p. } \delta \\ Y & \text{w.p. } (1-\delta) \end{cases}$$

Also, the following conditions are in order:

(1) the inequality in (2.5) is satisfied as an equality, and

(2) the mean and variance of D, μ and σ^2 respectively, are matched.

Condition (1) requires that

$$Var(Y) = 0$$

In other words, Y' takes values such that $Y = |Y' - Q|$ is a constant. Hence,

$$Y' = \begin{cases} Q+\beta & \text{w.p. } p \\ Q-\beta & \text{w.p. } 1-p \end{cases}$$

From condition (2), by equating the mean and variance of D, we obtain

$$\mu = (1-\delta)[p(Q+\beta) + (1-p)(Q-\beta)] \quad (2.12)$$
$$\sigma^2 = (1-\delta)p(1-p)(2\beta)^2 + \frac{\delta\mu^2}{(1-\delta)} \quad (2.13)$$

Solving (2.12) and (2.13) for β and p, we obtain,

$$\begin{aligned}\beta &= \left(\frac{\mu}{1-\delta} - Q\right)\sqrt{1+\phi} \\ p &= \frac{1}{2} + \frac{1}{2\sqrt{1+\phi}}\end{aligned}$$

where,

$$\phi = \frac{(\sigma^2(1-\delta) - \delta\mu^2)}{(\mu - Q(1-\delta))^2} \tag{2.14}$$

It is easy to verify that ϕ in (2.14) is strictly positive. Hence, both p and β are well defined. Thus, G^* is given by:

$$G^* = \begin{cases} 0 & \text{w.p. } \delta \\ Q+\beta & \text{w.p. } (1-\delta)p \\ Q-\beta & \text{w.p. } (1-\delta)(1-p) \end{cases}$$

Hence the proof. $\triangle$.

2.2 Optimal Order Quantity

Substituting (2.5) in (2.3) we obtain:

$$\Gamma^{G^*}(Q) = dQ + \frac{m+d}{2}\left[\delta Q + \sqrt{(1-\delta)(\sigma^2 + (Q-\mu)^2 - \delta Q^2)} + (\mu - Q)\right] \tag{2.15}$$

Now, consider the problem of finding the stock quantity, Q^* that minimizes $\Gamma^{G^*}(Q)$. It is easy to verify that Γ^{G^*} is strictly convex in Q. From the first order conditions, it follows that:

$$Q^* = \frac{1}{(1-\delta)}\left(\mu + k\sqrt{\frac{\sigma^2(1-\delta) - \mu^2\delta}{(1-\delta+k)(1-\delta-k)}}\right) \tag{2.16}$$

where,

$$k = \left(\frac{m-d}{m+d} - \delta\right)$$

If the Q^* in the above expression turns out to be negative, then the value of the order quantity, Q^* is set at zero. This is because, the function to be minimized (2.15) is convex with the constraint that $Q \geq 0$.

Here, note that $[\sigma^2(1-\delta) - \mu^2\delta] > 0$ and $1-\delta-k > 0$ always. Thus, the only case in which a real value of Q^* will not exist is when,

$$\begin{aligned}&1-\delta+k < 0 \\ &\Rightarrow \delta > \frac{m}{m+d}\end{aligned} \tag{2.17}$$

In this situation the solution is to be interpreted as $Q^* = 0$. Observe that this solution matches with the optimal solution for the case when the complete distribution of the demand is known.

2.3 Numerical Example

Consider a demand D with the following distribution with support on $S := \{0, 1, 2, \ldots, 8\}$:

$$P(D = i) = \begin{cases} 0.2 & \text{for } i = 0 \\ 0.1 & \text{for } 1 \leq i \leq 8 \end{cases}$$

The mark-up m and discount d are 1 and 0.5, respectively. The optimal solution from (2.4) of this problem is given by $Q = 5$ and the profit at this Q is $2c$ for any unit cost of purchase, c.

For the above distribution of demand, the mean, $\mu = 3.6$, the variance, $\sigma^2 = 7.44$ and $\delta = 0.2$. Under Scarf's rule, where only μ and σ are used $Q_s = 4$ and the profit at Q_s is 1.9c which is 5% less than the optimal, whereas under our rule, utilizing the δ we obtain $Q^* = 5$ and the actual profit as $2c$ which is also the optimal solution.

Now, had only μ, σ and δ been known and not the underlying distribution, under Scarf's rule the lower bound on profit would have been $1.69c$ at $Q_s = 4$ whereas according to our rule, the lower bound on profit is $1.8c$ at $Q^* = 5$.

3 Application to an M/G/1 Queue with Server Vacations

The analysis in the foregoing sections is more relevant in the context of inventory control models (where the newsboy problem appears indirectly) than in the context of the direct newsboy problem itself. In the newsboy problem, often it is not easy to guess the probability that the demand is zero, and even if an educated guess is possible, since the mean is much larger than zero in general, this probability will not affect the Scarf's ordering rule substantially. In contrast, in some inventory control problems of manufacturing systems, the relevant probability can be easily computed and can be used to get tight bounds on system wide costs. We cite an example here.

Consider the case of multi-item production-inventory systems, where the problem is to find optimal base-stock levels for a given production sequence of items. Many a time, it is only possible to derive moments of demand over lead time. With the knowledge of moments, distribution free procedures can be applied to find approximate base stock levels for items. For further details, refer [2] . Since in such multi-item systems, the base model is an M/G/1 queue with server vacations, for illustration purposes, we describe here a simple production-inventory system which can be modeled as a queue with server vacations.

Consider a production system where demand for a unit of the item comes according to a Poisson process with rate λ. Items are maintained in an inventory up to a base-stock level R. Assume that production times for individual units of item are *i.i.d* with cdf $S(.)$ and mean μ. Let, $\rho := \frac{\lambda}{\mu} < 1$. If a demand cannot be met from the inventory, it is backlogged. The following costs are incurred:

- C_b: the backlogging cost per unit per item
- C_h: the inventory carrying cost per unit per item

Once the production is initiated, the facility continues its production until the inventory level reaches the base-stock level R, (*exhaustive* service), and then gets engaged on some other production activity which lasts for a random duration (hereafter, referred to as *vacation* period). Assume that the vacation period has finite mean and variance. If one assumes that $(S-1, S)$ replenishment for requests come from the inventory, then the dynamics of the system can be modeled as $M/G/1$ queue with server vacations. If the facility returns after vacation for production and finds no requests in the order queue, then it goes for another vacation. Given the above cost structure, there is a trade off between excessive inventories and large backorders. Assuming stability, the long-run average cost incurred is given by:

$$TC = C_b E(L-R)^+ + C_h E(R-L)^+ \tag{3.1}$$

where L is the steady state shortfall, defined as

$$L = R - I$$

where, I is the steady state inventory level. A problem of interest is to find the optimal base-stock level R^* which minimizes (3.1). Observe that (3.1) has a structure similar to the newsboy problem with L as the demand distribution. In order to compute R^*, the complete knowledge of the distribution of L is required. It is not easy to find the complete distribution of L. However, it is possible to determine the mean and variance, μ_L, σ_L^2 respectively, of L, and the steady state probability that L is zero δ_L.

Define cycle time, C, as the time from the end of a service session (before the vacation starts) to the end of another service session. Using regenerative process arguments, the cycle time can be written as:

C = time until the first arrival + the remaining vacation period + the length of a busy period.

The steady state expected values of the above terms are related by:

$$E[C] = \frac{1}{\lambda} + E[V_e] + E[B] \tag{3.2}$$

where, V_e is the (equilibrium) excess distribution of V, and its expected value is given by,

$$E[V_e] = \frac{E[V^2]}{2E[V]}$$

Also, from the stability condition, it follows that,

$$E[B] = \rho E[C] \tag{3.3}$$

$$\delta_L = \frac{1}{\lambda E[C]}$$

Hence,

$$\delta_L = \frac{1-\rho}{1+\frac{2\lambda E[V]}{E[V^2]}} \tag{3.4}$$

One can use (3.4) to compute bound on (3.1) using (2.5) and the worst case optimal base stock level R^* using the analysis of the previous section.

Remark:

Note that for reasonable values of ρ, say ranging from 0.3 to 0.6, the optimal shortfall from (2.16) can sharply deviate from the optimal value derived from the Scarf's rule. Also, interestingly, in any multi-class queueing system, system utilization for each individual class will in general lie in the range mentioned above and hence, the value of δ will drastically affect the lower bound on objective value.

4 Conclusion

In this paper, we have discussed a variant of the distribution free newsboy problem with an additional knowledge on the probability of demand assuming the value zero. A lower bound on the objective and the distribution which achieves the bound have been presented. The bound presented here can be used in many inventory control problems where newsboy problem is embedded. For example, in a multi-item production-inventory system, a problem of interest could be determining periodic sequence for production under base-stock policies. Based on the above bounds, it is possible to perform approximate analysis and arrive at a near-optimal sequence. Further, for such systems, the above bounds can also aid in evaluating various static policies with regard to their worst case performance.

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