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Optimal Dynamic Production and Sales Plans for Innovations in the Presence of Scarcity Effects

By

Rahul Patil
Prof. Janat Shah

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Please address all your correspondence to:

Rahul Patil
Supply Chain Management Center
Indian Institute of Management Bangalore
Bangalore – 560 076
Email: rahul.patil@iimb.ernet.in

Prof. Janat Shah
Indian Institute of Management Bangalore
Bangalore – 560 076
Email: janat@iimb.ernet.in

Author: *Rahul Patil, Janat Shah*
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Rahul Patil
Supply Chain Management Center, Indian Institute of Management, Bangalore, India-560076, rahul.patil@iimb.ernet.in

Janat Shah
Indian Institute of Management, Bangalore, India-560076, Janat@iimb.ernet.in

Abstract
In this paper, we develop an integrated production and sales planning model for a firm that wants to sell an innovation with a fixed market potential under a supply constraint where a new product diffusion process depends upon word of mouth effects of waiting (lost) applicants in addition to adopters and innovators. This allows us to represent scarcity effects and its consequences namely hype and retarding effects. Under a general complete backorder situation, we show that myopic sales plan that sales as much as possible is a dominant policy when the hype effect exists. We also identify situations where the strategic sales delay plan can be optimal. We then compare our results with the prior research under a stylized lost sales setting. We prove that myopic (build up) plans are optimal in the presence of the hype (retarding) effects. Furthermore, the length of the build up period depends upon the magnitude of the retarding effect. We conduct an extensive numerical study which suggests that sales and production plans, profits and capacity sizing decisions strongly depend upon the response of waiting (lost) applicants.

Key words: Innovation diffusion models; integrated production and marketing decisions; optimal control theory; capacity planning

1. Introduction

Nowadays, innovations with shorter life cycles are becoming increasingly common in several industries. In toys and apparel industries, the short life cycle can be due to fashion effects. Sometimes, the decrease is a consequence of the speed of innovation-as in high technology industry (Kurawarwala and Matsuo 1996). For many such products, it is important to successfully manage an integrated operations and marketing process to sustain competitive advantage (Shah 2004). The importance of this issue is well recognized by academic researchers (Mahajan et al. 2000). One stream of this research has analyzed supply constrained diffusion dynamics of the innovation. Recently, Ho et al. (2002) and Kumar and Swaminathan (2003) developed models to study demand and sales dynamics of the diffusion process of the innovation under a supply constraint. To incorporate supply constraint effects, they suggested modifications to the original Bass model (Bass 1969), which assumed that only the customers who had successfully purchased the product and not the customers who had demanded the product controlled the future product growth. Thus, their models assumed that waiting or lost customers did not directly influence the subsequent product growth.

However, there are many situations where the demand for the new product indeed depends upon the behavior of waiting (lost) customers. Sometimes, the response of the waiting (lost) customers can accelerate the product growth. For example, in 1983, the shortage of soft sculpted dolls further fueled its demand and created the Cabbage Patch Panic (Langway, Hughey, McAlevey, Wang, and Conant 1983). Similarly, for its Wii game console, Nintendo has experienced a sharp increase in the demand due to supply shortages (Business Week 2007, Knowledge @Wharton 2006). Such customer behaviors are referred as hype effects where the scarcity of the product encourages customers to buy sooner.

On some other occasions, the unavailability of the product can slow down the product growth. For example, Jain et al. (1991) studied the diffusion of new telephones in Israel and found that the waiting customers tend to communicate negative information about the product which subsequently resulted in the slower growth. We call this as a retarding effect where the scarcity of the product discourages immediate purchases. In this paper, we use 'scarcity effect' to represent a supply constrained diffusion process dynamics. The consequences of the scarcity effect- the demand acceleration (deceleration) are referred as the hype (retarding) effects respectively. Our research is motivated by these contrasting customer behaviors in the presence of the product shortages.

Many strategic operations planning decisions related to the new product assume that the demand process is both stationary and time invariant (Silver, Pyke and Peterson 1998). Product diffusion patterns however interact with several operations planning decisions such as production, inventory and capacity planning in addition to sales (Cohen et al. 2000). Given that the scarcity effect can significantly affect the shape of the product diffusion process, we believe that there is a need to include it while developing integrated sales and production plans for the new product. The insights obtained from such research can help practicing managers to address the following questions: How should a marketing manager design a sales strategy in the presence of the hype and retarding effects? For an operations manager, is it a good idea to set capacity by anticipating a particular scarcity effect?

In this paper, we characterize dynamic optimal production and sales plans for the new product under a supply constraint by explicitly including the response of waiting (lost) customers. We use a parsimonious supply restricted diffusion model (Jain et al. 1991) to represent the scarcity effects. We first show that optimal sales plan strongly depends upon the waited customer's response. We then compare our results with the prior research under a stylized lost sales situation. First, we show that the myopic policy is optimal when the hype effect exists. Kumar and Swaminathan (2003) proved the optimality of the build up plan under this setting without including scarcity effects. Thus, our results suggest that the optimal sales structure can change significantly when the firm includes the hype effects in the optimization model. Then, we show that the build up policy is optimal in the presence of the retarding effects. Furthermore, the length of the build up period depends upon the magnitude of the retarding effect. For the general problem with complete backordering, we identify situations where the myopic and strategic sales delay policies are optimal. Theorem 1 characterizes the behavior of optimal policies for the general problem. We numerically solve the problem via discrimination to study the effects of the parameters on the optimal profits, sales policies and build up periods. Dominant policies suggested by Ho et al. (2002) and Kumar and Swaminathan (2003) contrast each other. We finally use an envelope theorem and attempt to explain the reasons behind this.

The rest of the paper is organized as follows. Section 2 reviews the relevant literature. The mathematical model formulation is presented in the Section 3. Section 4 characterizes the optimal products and sales plans and shows that the myopic policy is always an optimal policy when the influence of the waiting customers is the strongest. Section 5 reports the set up and the results on the numerical study. The paper ends with conclusions and the directions for future research.

2. Motivation and Literature Review

First, we report the research where the demand for a new product occurs as a diffusion process. In an innovation diffusion process, an innovation is first accepted by innovators via mass media, who then influence imitators to adopt the product by means of interpersonal communications including word of mouth and non verbal observations (Bass 1969). Bass (1969) has proposed a parsimonious mathematical diffusion model using the theory of communication where the diffusion process can be represented using the following differential equation. $z(t)=\left\{p+\frac{q_1}{m}S(t)\right\}[m-S(t)]$ where $z(t)$ is instantaneous demand at time t, p and q_1 are constants $\in (0,1)$, which represent the effects of mass media and adopters on potential adopters respectively. S(t) is the cumulative sales at time t while m is the market potential. The Bass model is a theoretically sound and an empirically robust model, and therefore is widely used in both practice and academics to develop both analytical and empirical models (Krishnan and Jain 2006). The marketing science literature has often used it to compute optimal pricing and advertising levels for the new products. Mahajan et al. (2000) provide an excellent review on this literature.

Despite of the valuable contributions, from an operations management perspective, the fundamental limitation of the Bass model is its inability to model supply restrictions. For example, Motorola (supplier of G4 chips) was not able to meet the rapid growth of demand of Apple's new PowerMac G4 (New York Post 1999). Surprisingly, there are only two published papers that have attempted to analytically derive the optimal dynamic sales and production plans in the presence of supply constraints (Ho et al. 2002, Kumar and Swaminathan 2003). Ho et al. (2002) analyzed a supply constrained diffusion process by including inventory holding and lost sales costs. They showed that the myopic policy is always optimal under supply constraints. They also found that the strategy of delaying a product launch can be optimal in some situations and that optimal time to market and capacity increase with both the coefficients of innovation and adoption.

Kumar and Swaminathan (2003) included backorder costs in addition to inventory and lost sales costs and showed that the build up policy is optimal in the lost sales setting. In an extensive numerical study, they found that the build up plan is dominant in contrast to Ho et al. (2002) results which favored the myopic plan. This contradiction is intriguing. Kumar and Swaminathan (2003) note that although the models used in both papers are different , it is unlikely that the differences in the models alone account for the contrasting results and hence resolving this issue is a matter of the further research.

These papers used the demand function, which did not consider the direct response of waiting applicants on the diffusion process which could be quite significant. According to Simon and Sebastian (1987), an empirically observed demand diffusion may be influenced by supply chain bottlenecks such as production capacity, distribution etc and therefore, natural demand process can accelerate or retard. Recently, the shortages of PlayStation 3, Sony's next generation game console have accelerated its demand. It was because of the hype effect aroused due to Sony's inability to produce in sufficient quantities to meet the demand (Business Week 2007). iPhone and final Harry Potter book are some other examples where the hype effect has accelerated the demand growth (Quelch 2007). Economic, psychological and sociological literatures provide several explanations to this phenomenon. E.g. the possession of an unavailable product provides a valued sense of self uniqueness (Synder and Fromkin 1980). The possession and display of an unavailable product is a source of status (Veblen 1965). On the other hand, as discussed in the introduction section, the scarcity effect can also retard the diffusion process which is referred as the retarding effect.

Jain et al. (1991) proposed an enhancement to the Bass diffusion model to study the scarcity effects and found that the diffusion process could take different shapes depending upon the response of the waiting applicants. Because the focus of their research was on empirically testing the enhancement, they did not analytically investigate the influence of the hype and retarding effects on the optimal sales and production policies. Based on the insights obtained from the prior research on integrated production and sales planning for the new products, we believe that the optimal sales and production plans, capacity level, and profits will depend upon these effects. In this paper, we extend this literature by presenting a general analytical model to characterize the optimal sales and production plans for the innovations in the presence of the scarcity effects.

3. Mathematical Model Formulation

Consider a firm that wants to decide production and sales plans for its new product. We assume that the firm can start the production of the product at a known date, which we define as t =0.

Let,

q_2 = influence of waiting applicants on potential adopters

s(t)= instantaneous sales (adopters) of the product at time t

a(t) = instantaneous backorders (lost sales) at time t

Z(t) = cumulative demand for the product at time t

A(t) = cumulative number of backorders (lost sales) at time t

Z(t) = S (t) + A(t)

z(t) = s(t) + a(t)

Under a supply constraint, the diffusion process can be described as follows. At time t, potential adopters who have not yet purchased the product may place an order. The demand is fulfilled if the product is available at time t; otherwise the customer either waits (backordering), or cancels her order (lost sales).

Jain et al. (1991) propose a modification to the Bass model (Bass 1969) to include these effects. The model is parsimonious, intuitive, and more importantly, empirically validated. Empirical validation is important because the lack of it can question the generalizability of the implications of the results (Krishnan and Jain 2006). Also, the model reduces to the Bass model when $q_1 = q_2$ which is desirable given the strengths of the Bass model. Hence, we use this diffusion model (equation 1) in our mathematical formulation. We assume that unconstrained diffusion process follows the classical S curve. Unrestricted q_2 allows us to model both positive and negative word of mouth of the unmet demand.

$$z(t) = \left\{ p + \frac{q_1}{m} S(t) + \frac{q_2}{m} A(t) \right\} \left[m - S(t) - A(t) \right] \qquad (1)$$

Figure 1 shows how our model represents both hype and retarding effects. In general, when $q_2 > q_1$, we observe the hype effects and when $q_2 < q_1$ we have retarding effect. Both Kumar and Swaminathan (2003) and Ho et al. (2002) diffusion models are a special case of our model when word of mouth effects of waiting customers is zero (q_2=0).

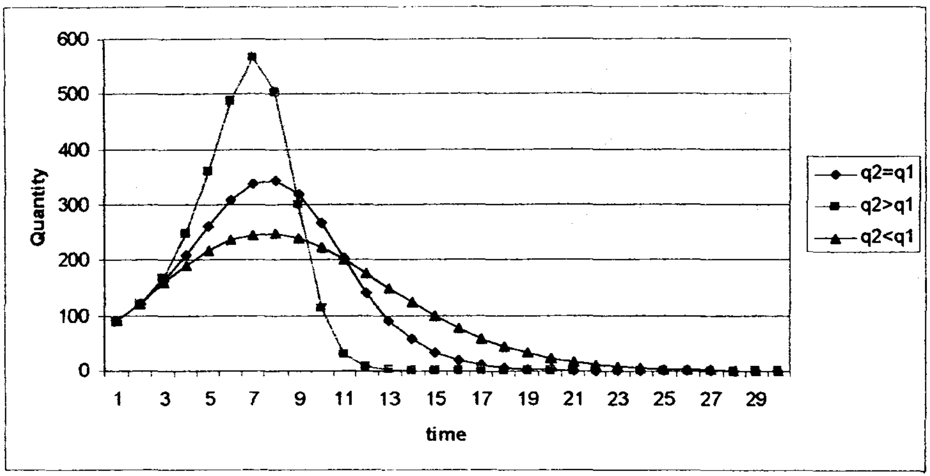

Figure 1: Hype and retarding Effects in the supply constrained diffusion process.

Let,

α = unit production cost,

π = selling price per unit;

w = back order cost per unit backlogged per unit time,

h = inventory holding cost per unit inventoried per unit time,

γ = discount rate

i(t) = inventory at time t

x(t)= production at time t

c = production capacity

The problem can be formulated as (P1) where the objective is to maximize the discounted profit of the

firm over the life cycle of the product by subtracting discounted inventory, backorder and production costs from discounted revenue (equation 2). The firm can manage the innovation diffusion process using equations (3-5). Instantaneous inventory and backorders follow equations (6-7).

$$(P1) \quad \max_{s(t), x(t), o \le t \le T} \phi = \int_{o}^{T} e^{-\gamma t} \{\pi s(t) - \alpha\, x(t) - w\, A(t) - h\, i(t)\} dt \tag{2}$$

subject to

$$\dot{Z}(t) = z(t) \tag{3}$$

$$\dot{S}(t) = s(t) \tag{4}$$

$$\dot{z}(t) = -\,pz(t) + \frac{q_1}{m}\{-S(t)z(t) + (m - Z(t))s(t)\} + \frac{q_2}{m}\{-A(t)z(t) + (m - Z(t))a(t)\} \tag{5}$$

$$\dot{i}(t) = x(t) - s(t) \tag{6}$$

$$\dot{A}(t) = a(t) = z(t) - s(t) \tag{7}$$

$$x(t) \le c \tag{8}$$

$$s(t) \ge 0 \tag{9}$$

$$A(t) \ge 0,, i(t) \ge 0, \quad A(0) = 0,\ i(0) = 0 \quad \textit{for all } t \ge 0 \tag{10}$$

It is difficult, if not impossible, to obtain the closed form solutions for the optimal production and sales policies for this problem using optimal control theory principles. Hence, we use a novel method to obtain the closed form solutions and to derive the characteristics of the optimal policies by using the properties of the growth curve and basic non linear optimization principles.

4. Analytical Results

LEMMA 1. *For every optimal sales plan, there exists a time* t_c *such that* *for all* $t \ge t_c, z^*(t) < c$

Proofs of all lemmas, propositions and theorems can be found in appendix.

Intuitively, this lemma implies that for a fixed market potential, during the decline phase, at certain point in time, the instantaneous demand has to decrease to and stay below the chosen capacity. During this time period, because it is not profitable to backlog the demand, the optimal sales are always higher than or equal to the instantaneous demand. This means that the lost sales and additional backorders are incurred before this time period.

Now, we analyze distinct diffusion patterns that the firm can observe depending upon the chosen capacity level c. Specifically, the product growth can exhibit two different regimes. In the first regime, production capacity is less than instantaneous demand up to time t_c. So, the diffusion process starts in a constrained phase and then switches to an unconstrained phase after time t_c. In the second regime, the diffusion process starts in an unconstrained region, which then switches to a constrained phase and finally returns back to an unconstrained region.

Note that a (t) consists of both unavoidable sales delays which occur when the capacity is less than the demand and strategic sales delays that are incurred to further control product growth. We define a strategic sales delay policy (SSD) as a policy that strategically delays the sales of available product units over the life cycle with the objective of maximizing profits. Myopic policy (MP) on the other hand never delays the sales, and thus, sales as much as possible. Let, $a_u(t)$ *and* $a_s(t)$ denote unavoidable and strategically delayed sales respectively. Note that $a(t) = a_u(t) + a_s(t)$ *and* $A(t) = A_u(t) + A_s(t)$. Thus,

Under MP, $a_s(t) = 0$ while under SSD, $a_s(t) > 0$.

(11)

We rewrite equation by including $a_u(t)$ *and* $a_s(t)$ terms.

$$\dot{z}(t) = -pz(t) + \frac{q_1}{m}\{-S(t)z(t) + (m - Z(t)s(t)\} + \frac{q_2}{m}\{-(A_u(t) + A_s(t))z(t) + (m - Z(t)(a_u(t) + a_s(t))\} \quad (12)$$

Because extreme scarcity effects ($q_2 \rightarrow \infty or -\infty$) can make the diffusion model (equation 1) infeasible, and so in this paper we restrict q_2 such that the diffusion process remains feasible under the strategic sales delay policy (with SSD of atleast 1 unit). Our analytical results are valid over the q_2 range (0, 1). Our extensive numerical experiments suggest that the q_2 range is wide enough to encompass almost all the practical situations.

For an instantaneous demand at time t, sales at time t depend upon the a(t). For example, when a(t) = z(t), then s(t) = 0. In other words, the firm is ultimately deciding the sales plan by selecting the number of waiting (lost) applicants at time t over the product life cycle. The firm can only control the strategically delayed sales for a given capacity. Hence, we reformulate the problem such that we have to only choose the optimal values for $a_s(t)$ and this will automatically define the optimal sales path. Some researchers and practitioners argue that the firms intentionally delay the sales to accelerate demand while others believe that it is a consequence of unavoidable sales delays due to supply restrictions (Business Week 2007, Knowledge@Wharton 2006). Our new decision variable will allow us to test the merit of these strategies in the case of the hype effects.

We now introduce and examine the stylized lost sales problem where assume γ = 0, h=0, w = 0, α=0 that is exactly similar to Kumar and Swaminathan (2003). First, this situation allows us to better understand the dynamics of the diffusion process under the scarcity effect. Second, we want to study whether Kumar and Swaminathan (2003) results hold in the presence of scarcity effects. Third, using the Lemma 1 property, the firm could minimize its total backorder (lost sales) costs by minimizing the cumulative backorders (lost sales) occurring before the time t_c. This implies that the policy that minimizes the total lost sales costs in the complete lost sales case also minimizes the total backorder costs in the complete backorder case which allows us to use the insights from the lost sales case to the general problem P1.

In this case of lost sales, for a fixed market potential, the profit maximization over the product life cycle is equal to the lost sales minimization over the life cycle of the product by choosing optimal production and sales policies. The optimal production plan is to produce units as soon as possible to not only meet the current demand but also to create buffer for the future as the inventory holding cost is zero. The optimal sales plan though not trivial can be computed by solving the following non liner optimal control problem (P2) as shown below.

$$(P2) \quad \min_{a_s(t) o \le t \le \infty} \lambda = Z(\infty) - S(\infty) \tag{13}$$

$$\textit{subject to}$$

$$\textit{equations}: 3, 4, 5 \textit{and}$$

$$0 \le s(t) \le z(t) \tag{14}$$

$$S(t) \le ct, \tag{15}$$

$$A(t) \ge 0 \textit{ and } i(t) \ge 0 \quad \textit{for all } t \ge 0 \tag{16}$$

$$A(0) = 0, i(0) = 0 \tag{17}$$

Now, we characterize the behavior of the optimal policies in the presence of hype effects for both the problems P1 and P2.

4.1 When $q_2 > q_1$ (hype effect)

Lost Sales Case (Problem P2).

In the lost sales case, the SSD policy results in the strategic lost sales whenever the firm delays the sales of the available units.

$\theta = \int_{t_c}^{\infty} z(t)dt$ indicate the total demand that occurs after time t_c. From the Lemma 1, because the lost sales occur before t_c, the total lost sales (objective function) over the product life cycle can be redefined as :

$$\lambda = \int_0^{t_c} z(t)dt - ct_c \Rightarrow m - \theta = \int_{t_c}^{\infty} z(t)dt - ct_c \qquad (18)$$

For a given c and m, λ is decreasing in both t_c and θ.

Now, we show that this objective function (equation 18) also holds for regime 2. Note that the inventory in the unconstrained phase can be used to meet some demand in the constrained phase. Also, by definition, there exists an interval ($t_1 \le t \le t_c$) in the constrained region where i(t) + c(t) < z(t), and hence, the firm incurs unavoidable lost sales in this interval. So, objective for regime 2 is fundamentally the same as regime 1 objective (equation 18).

Let, t_c^m *and* t_c^b refer to the times during the decline phase when the instantaneous demand z(t) is equal to the capacity level c under MP and SSD respectively.

LEMMA 2. When $q_2 > q_1$, $t_c^m > t_c^b$ *and* θ *under SSD* < θ *under MP.*

PROOF. see Appendix.

Lemma 2 shows that both t_c and θ are lower under strategic sales delay policy compared to the myopic policy. Figure 2 graphically shows how t_c decreases as strategic sales delay increases. Drop down lines show the t_c associated with each *SSD* level when c = 150. These properties are the consequences of the diffusion process. Demand accelerates when the firm implements the strategic sales delay policy, which subsequently shortens the time needed to bring the instantaneous demand to the capacity level.

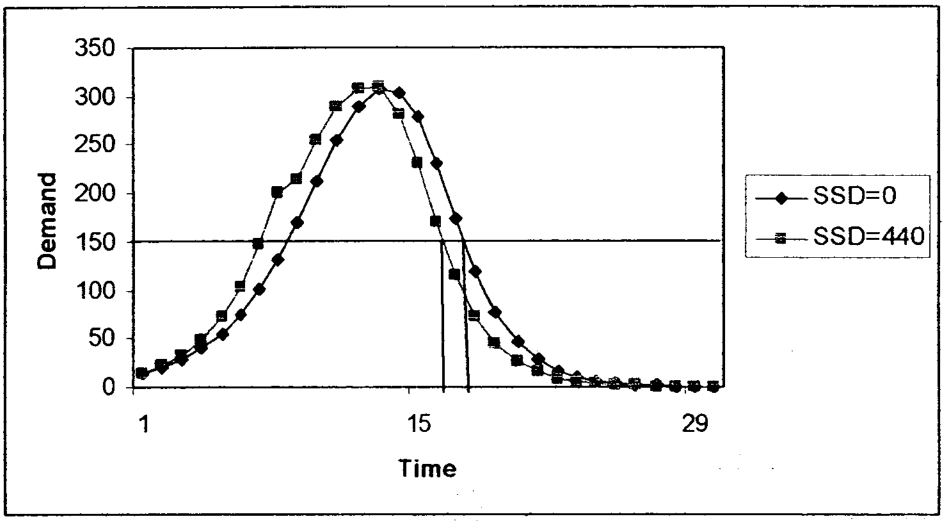

Figure 2: t_c as a function of strategic sales delay when q_2 is positive and higher than q_1 (p=0.005 q_1=0.4, q_2=0.5, m =3000, T= 30, C=150, complete backordering).

$\frac{\partial \lambda}{\partial \theta} < 0$ *and* $\frac{\partial \lambda}{\partial t_c} < 0$. According to Lemma 2, $\frac{\partial \theta}{\partial a_s(t)} < 0$ *and* $\frac{\partial t_c}{\partial a_s(t)} < 0$ *for all* $t \geq 0$

Therefore using chain rule, $\frac{\partial \lambda}{\partial a_s(t)} > 0$ *for all* $t \geq 0$

Therefore, λ is minimized when $a_s^*(t) = 0$ for all $t \geq 0$. Under the stylized lost sales situation, Kumar and Swaminathan (2002) suggested that the SSD was an optimal plan. Their model assumed that the product shortages generated only a specific kind of retarding effect. But, when we include the true scarcity effect (hype effect) in the stylized setting, then our results suggest that SSD is never an optimal sales plan. Clearly, the optimal sales plan can change drastically depending upon the scarcity effects.

Now, we use the insights obtained from this exercise for problem P1 to prove our proposition 1.

PROPOSITION 1. *For the optimization problem (P1), in the regime 1 and in the regime 2 (when* $\gamma = 0$), in the presence of the hype effects, *myopic sales plan is always an optimal policy. That is.*

$$s^*(t) = \begin{cases} z^*(t) & \text{if } A^*(t) = 0 \text{ and } i^*(t) > 0 \\ x^*(t) & \text{if } A^*(t) > 0 \text{ and } i^*(t) = 0 \\ \min(z^*(t), x^*(t)) & \text{if } A^*(t) + i^*(t) = 0 \end{cases}$$

(19)

And in the remaining regime 2, the optimal sales plan is given by Theorem 1.

Proposition 1 suggests that in the presence of the hype effects, the strategic delay of the sales (which only results in the product growth acceleration) is never an optimal policy whenever the firm is capacity constrained from the beginning or when the discounting factor is very low in the regime 2. This is because the hype effect resulting from the SSD only increases backorder and inventory holding costs without increasing the discounted revenue. On the other hand, the MP that results in reduced backorders and inventory holding costs without reducing (sometimes increasing) revenue due to immediate sales is an optimal plan. Consistent with this prescription, after facing the supply shortage and the resulting hype effect, both Sony and Nintendo have used costly options such as airfreight to keep the retailer shelves full (Business Week 2007). Figure 3 depicts the structure of the myopic strategy where the firm sells equal the instantaneous demand till period 3 followed by the sales that equal capacity till period 11 and finally selling as per the instantaneous demand.

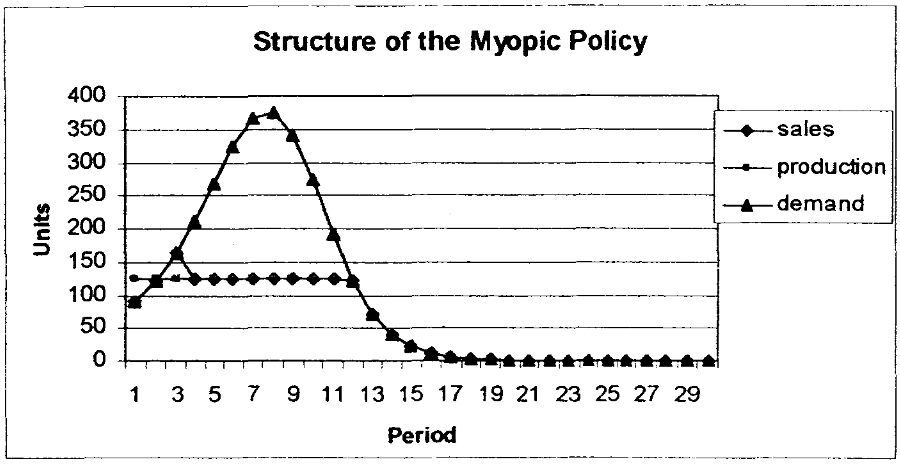

Figure 3: Structure of the myopic policy (p=0.03 and q_1=0.4, q_2=0.5, m =3000, T= 30, C=125, h=0.001, w=0.01 and β =0.995, lost sales).

Theorem 1 (introduced in the next section) allows for the strategic sales delay policy in the remaining part of regime 2. We anticipate some situations where this policy can be optimal. For example, when $\gamma > 0$, h=0, w = 0, c= high, p=0, $q_2 >> q_1$, then because Z(t) under SSD > Z(t) under MP for all t >0, discounted revenue under SSD would be higher than MP. Therefore, in this situation, it would be profitable for managers to intentionally delay the sales.

We conducted a small numerical study by keeping h, w, p at low and high levels (as described in the section 5) and γ and c at three levels (low, medium and high) and the rest of the parameters were as shown in section 5. We found that though the myopic strategy was dominant (53 out of 72 times); the SSD became dominant whenever the capacity and discounting were high and inventory and the mass media influences were low. When the mass media influence was low, the diffusion process was slow. And, when discounting were high, it was profitable for the firm to accelerate the demand as long as it had capacity to manage the acceleration, which resulted in the optimal SSD policy.

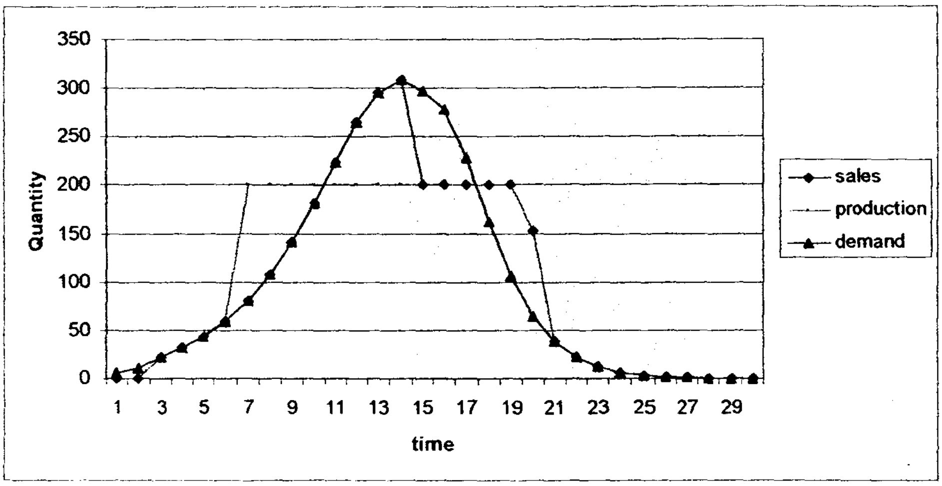

Figure 4: SSD policy in the presence of the hype effects.

Figure 4 shows the structure of the SSD policy which intentionally delays the sales. The sales delay occurred during the initial periods. This was evident from the diffusion process (equation 1 and 12) where the strong influence of the early sales delay was felt not only in the period it was incurred but also in the future periods as well through the diffusion effects. Because the optimality of the SSD depends upon the cost structure, chosen capacity level and the discounting factor, the manager should first analyze the market situation and then make the important decision of delaying the sales.

4.2 When $q_2 < q_1$ (retarding effect)

In this section, we generalize Kumar and Swaminathan (2003) results under lost sales setting and show that the build up policy is optimal whenever the firm experiences the retarding effect. Build up plan entirely looses the sales occurring between (0, t_s^*) and then never looses any sale. Furthermore, the build up period $(0, t_s^*)$ can take different shapes and can decrease as the lost customers communicate more negative word of mouth about the product.

PROPOSITION 2. *For the problem P2, If the product demand follows the Bass model with a supply constraint and when* q_2 is less than q_1,

When $z^*(t_s^*) > 0$; *the optimal sales plan is a build up policy:*

$$s^*(t) = 0 \qquad t \le t_s^*$$
$$s^*(t) = z^*(t) \qquad t \ge t_s^* \le \infty$$

(20)

And optimal t_s^* *is such that* $A_s^*(t_s^*) = A_u^*(T) - A_u^*(t_s^*)$

Otherwise the optimal sales plan follows theorem 1 with strategic lost sales occurring between $(0, t^*)$ *such that* $z^*(t^*) = \varepsilon$ *where* ε *is a very small positive number.*

CORROLORY 1.When $q_2 \le 0$, the BU period strictly decreases when q_2 decreases.

PROOF. Simply follows from the analysis of equations 1 and 12.

Figure 5 shows how the build up period decreases as the response of the waiting customers becomes harmful. We assume p= (0.001,0.03, 0.2) and q_1=0.6, m =3000, T= 30, C=125, h=0.001, w=0.01 and β=1. We assumed a linear demand between the periods to compute the build up period in a continuous time. The Figure 5 suggests that as q_2 decreases and becomes negative; the firm can not afford to delay the sales for a long time because such strategy can erode the profits due to slow product diffusion.

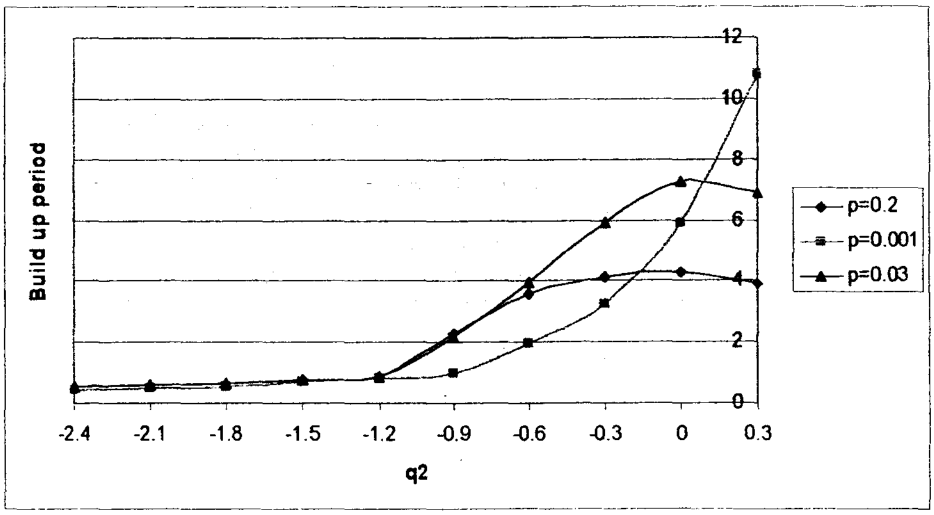

Figure 5: Build up period as a function of q_2 for different values of p and q_1.

Figure 6 shows that structure of the build up policy which does not sell a single unit until period 6 and then sells as per the instantaneous demand for the rest of the product life.

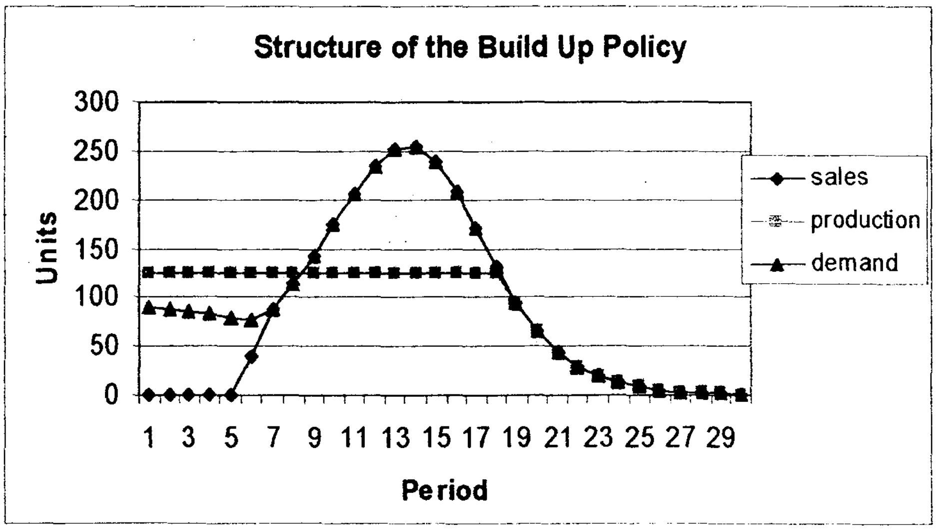

Figure 6: Structure of the build up policy (p=0.03 and q_1=0.4, q_2=0, m =3000, T= 30, C=125, h=0.001, w=0.01 and β =0.995, lost sales).

The following theorem illustrates the characteristics of the optimal sales and production plans for the general problem P1. We use the Pontryagin's maximum principle to prove theorem 1.

THEOREM 1. For all $t \geq 0$, the optimal sales plan for problem P1 can be described using the following form

$$s^*(t) = \begin{cases} 0 & if\ a^*(t)i^*(t) > 0 \\ z^*(t) & if\ a^*(t) = 0\ and\ i^*(t) > 0 \\ x^*(t) & if\ a^*(t) > 0\ and\ i^*(t) = 0 \\ \min(z^*(t), x^*(t)) & if\ a^*(t) + i^*(t) = 0 \end{cases} \quad (21)$$

Note that the proposition 2 is a special case of the theorem 1 where the coefficient associated with $s^*(t)$ is negative till t_s^*, and after t_s^*, it becomes and always remains positive. Thus, intuitively, the negative coefficient represents the value associated with delaying sales at time t. Figure 7 graphically illustrates this idea in the lost sales situation where each strategic lost sale results in the reduction of unavoidable lost sales (of more that one unit) and hence total lost sales up to the optimal point. Also, the value of strategic sales delay increases as the demand increases during the growth phase.

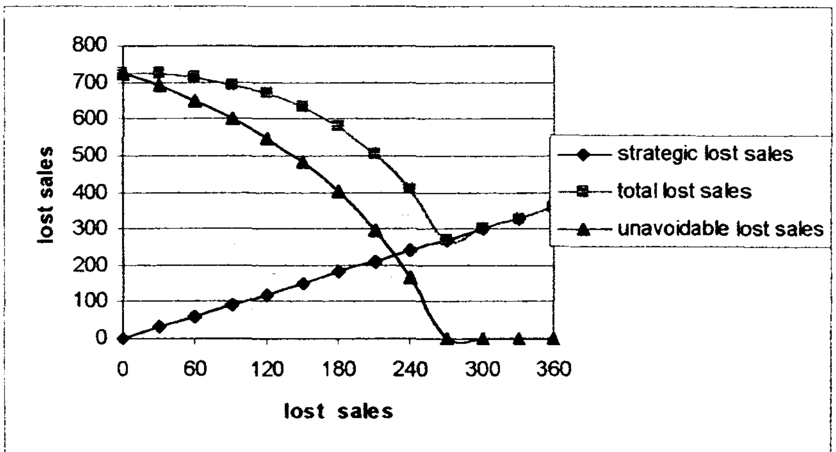

Figure 7: Structure of the build up policy (p=0.03 and q_1=0.4, q_2=-0.2, m =3000, T= 30, C=125, h=0.001, w=0.01 and β=1, lost sales).

5. Numerical Study

We conduct a numerical study to demonstrate the importance of including the scarcity effects while determining the optimal production and sales strategies. First, we show how optimal sales policy structure changes with q_2. Subsequently, we show that profits increase when the firm considers the q_2 effects. Finally, we analyze the impact of q_2 on the capacity planning decisions. We use the results of both propositions and theorem 1 to verify the optimality of the solutions obtained during the numerical study.

We discretize the problem (P1) as shown below to numerically compute the optimal strategies using non linear programming. In this formulation, we set N such that the market potential is almost entirely exhausted. We also convert γ to $\beta \in (0,1)$ for the discrete case.

$$(P3)\quad \max_{s(t),\,x(t),\,t=0,\ldots,N} \lambda = \sum_{t=0}^{N} \beta^{t}\{\pi s(t) - \alpha\, x(t) - w\, A(t) - h\, i(t)\}$$

$$\begin{aligned}
&\textit{subject to}\\
&Z(t+1) - Z(t) = z(t)\\
&S(t+1) - S(t) = s(t)\\
&z(t) = \{p + \frac{q_1}{m} S(t) + \frac{q_2}{m} A(t)\}\left(m - S(t) - A(t)\right)\\
&i(t) = \sum_{u=0}^{t} x(u) - S(t)\\
&A(t+1) = A(t) + z(t) - s(t)\\
&A(t) \ge 0,\ \textit{and} \qquad A(0) = 0\\
&i(t) \ge 0,\ \textit{and} \qquad i(0) = 0\\
&s(t) \ge 0\ \textit{and}\ \ x(t) \le c \qquad \textit{for all}\ \ t \ge 0
\end{aligned}$$

It is important to set parameters of our discrete model at appropriate values. Based on the analysis of 213 data sets drawn from different industries, Sultan (1990) reported that the mean values for parameters p and q_1 were 0.03 and 0.38 respectively. We use p=0.03 and q=0.4 in our experiments. We set m =3000, T= 36(sufficient time period to exhaust the market demand) and C = 125. If each period is of two weeks, then the corresponding product life cycle is around 1.5 years. We set per unit production cost at 1 unit and selling price per unit at 1.3 units.

We used an interval branch and bound algorithm to find the globally optimal solution to problem P3. The interval branch and bound algorithm processes a list of boxes that consist of bounded intervals for each decision variable, starting with a single box determined by the user specified bounds. On each iteration, it seeks lower and upper bounds for the objective and the constraints in a given box that will allow it to discard all or a portion of the box by proving that the box can not contain feasible solutions, or that it can not contain objective function values better than the known best bound. Boxes that cannot be discarded are subdivided into smaller boxes, and the process is repeated. This iterative process finally results in the boxes that enclose locally optimal solutions, and the best of these provides the globally optimal solution (Premium Solver Platform Version 7).

5.1Optimal Sales Strategies

To investigate the effect of inventory holding and backorder costs, we set 'h' at two levels (low: 0.001, high: 0.01) and 'w' at two levels (low: 0.001, high: 0.01). Discounting parameter β was set at two levels (low: 1, moderately high: 0.995). We consider lost sales and complete backlogging situations. For the lost sales case, we add the constraint 14 to P3. This resulted in 16 scenarios. For each scenario, we consider the following levels for q_2(0.5, 0, and -0.5). Table 1shows how the optimal sales plan changes when the firm includes waiting customer's effects. In the Table, M=Myopic, BU=Build Up, and number in the bracket denotes the build up period.

Although all the problem characteristics (except q_2) are the same, Figures 3,6 and 8 demonstrate how the optimal sales structure changes from myopic-build up- hybrid policies with q_2. In this problem situation, note the strategy shift from BU to M-BU-M. This finding can be explained as follows.

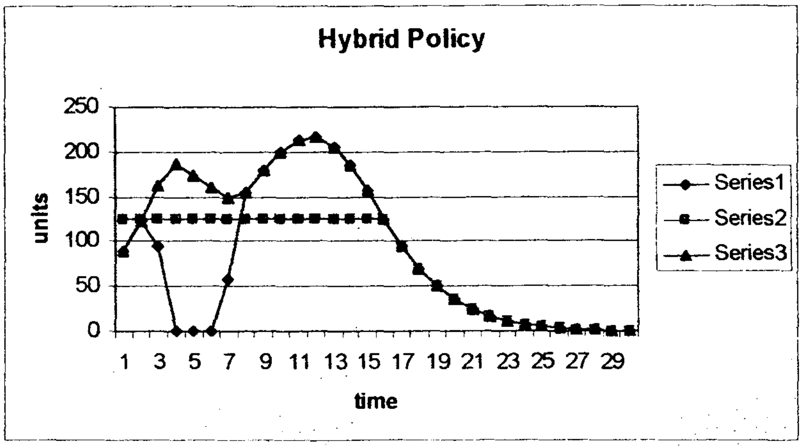

Figure 8: Structure of the SSD policy (p=0.03 and q_1=0.4, q_2=-0.5, m =3000, T= 30, C=125, h=0.001, w=0.01 and β =0.995, lost sales)

Effectiveness of BU strategy also depends upon the discounting factor in addition to lost sales, which indicates the magnitude of reduction in revenues arising from future periods-lower value implies the lesser value of future revenues. Because discounting is moderately high, initially the main focus is on increasing the sales that makes myopic policy dominant. But, it is also important to reduce the lost sales over the product life cycle to increase total revenues. Therefore, after certain point in time (as sales accelerate), the firm seeks to reduce the lost sales after being overwhelmed by the demand growth. Hence, the firm shifts to the BU plan for some time. Finally, as the firm builds inventory and also decelerates the demand growth, the firm shifts to the M strategy. Note that the firm is able to effectively control the demand acceleration during the growth phase because of the negative q_2 effects. Thus, depending upon the dynamics of the diffusion process in the presence of scarcity effects and the combination of the problem parameters, the optimal sales policy can take different shapes.

When inventory holding costs and discounting are low and back order costs are high, both in the lost sales and the complete backorder situation, the build up plan is dominant when $q_2 < q_1$. However, as inventory holding costs increase, the myopic policy becomes dominant. This shift can be explained using an envelope theorem as below (Klein 1998).

	h	w	β=1			β=0.995		
			q2=0.5	q2=0	q2=-0.5	q2=0.5	q2=0	q2=-0.5
Lost Sales	low	low	M	BU(5)	BU(2)	M	BU(5)	M-BU(3)-M
	high	low	M	M	M	M	BU(5)	M
	low	high	M	BU(5)	BU(2)	M	BU(5)	M-BU(3)-M
	high	high	M	M	M	M	BU(5)	M
Complete Back Order	low	low	M	M	BU(6)	M	M	M
	high	low	M	M	M	M	M	M
	low	High	M	BU(8)	BU(6)	M	BU(8)	M
	high	High	M	M	M	M	M	M

Table 1: Behavior of optimal sales strategies under different situations.

CORROLARY 2. When γ=0, in both the problems P1 and P2, for a given w, there always exists a critical 'h' above which the optimal sales plan is always MP in the presence of the retarding effects.

PROOF. Note that when h = 0, BU policy is optimal. Also, at optimality as per the envelope theorem,

$$\frac{\partial H(x^*, s^*, \pi, \alpha, h, w, \lambda)}{\partial w} = -\int_0^T a^*(t)dt \quad (22)$$

$$\frac{\partial H(x^*, s^*, \pi, \alpha, h, w, \lambda)}{\partial h} = -\int_0^T i^*(t)dt \quad (23)$$

The build up policy minimizes total back order costs while the myopic plan minimizes total inventory holding costs. For any per unit backorder cost, as per unit inventory holding costs increase, the equation 23 (which suggests the impact of reducing total inventory holding costs) becomes more governing than 22 and hence, there exists a critical value of per unit inventory holding cost, after which the optimal sales plan switches from the build up to the myopic. Unlike Kumar and Swaminathan (2003), Ho et al. (2002) does not include backorder costs. Hence, equation 23 is always dominant compared to equation 22. This may be the reason why the myopic policy is always dominant in their case.

5.2 Impact on the Optimal Profits

Let $\phi_{q2}^* \; and \; \phi_0^*$ denote the optimal profits and profits under the assumption that $q_2 = 0$ respectively. We then compute the percent improvement in profit as $\frac{\phi_{q2}^* - \phi_0^*}{\phi_0^*} x100$. When h=0.001, w=0.001, $\beta = 1$, p=0.03, q_1=0.4 when q_2=0.5 under the lost sales case, $\phi_{q2}^* = 494.7 and \; \phi_0^* = 466.4$ which resulted in the significant profit increase of 4.4 %. While when q_2=-0.5, $\phi^*_{q2} = 854 \, and \, \phi^*_0 = 846$ and the resulting profit increase was 0.935%. Clearly, the profits increase when the firm includes word of mouth effects of unmet demand. Note that the profit increase was higher when q_2=0.5 than when q_2=-0.5 because the radical change in the structure of the sales policies.

5.3 Capacity Planning Decisions

For the short life cycle products, capacity decisions are irreversible because of long lead time needed to adjust the existing capacity (Ho et al. 2002). Often, in high tech industry, capacity shortages result due to due to long capacity lead times. Therefore, Cohen et al. (2000) suggest that there should be an explicit link between the pattern of the product diffusion and the capacity planning decision. Now, we demonstrate the impact of hype and retarding diffusion patterns on the optimal capacity levels.

We set h=0.001, w=0.01, $\beta = 1$, p=0.03, q_1=0.4 and complete backordering, and q_2 at three levels (0.5,0,-0.5). We assume that per unit cost of adding the capacity is 1.5 unit while selling price is 1.3 units and production cost is 1 unit. Figure 9 shows that optimal capacity sizing decision can be different depending upon the influence of the waiting customers. Optimal capacity levels increase (decrease) as q_2 increase (decrease).

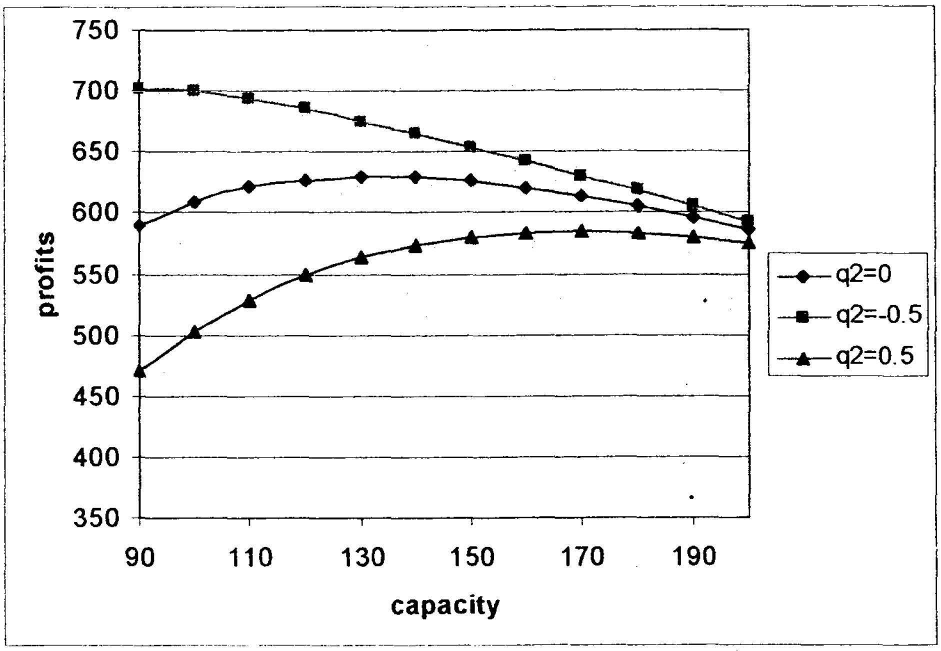

Figure 9: Optimal capacity levels as a function of q_2.

When $q_2 > q_1$, when shortages occur, due to scarcity effect, the demand accelerates which in turn generates additional shortages. The end result is higher backorders costs. Hence, the firm needs additional capacity (compared to the case when q_2=0) to dampen the resulting damage. This might be the reason behind Nintendo's decision to considerably increase its production capacity for its Wii console (Business Week 2007). But, when q_2 is negative, the demand decelerates when the shortage occurs, which allows the firm to not only decrease immediate shortages but also to build inventory(sometimes) to avoid future shortages. Therefore, the firm can operate at lower capacity compared to the base case.

6. Managerial Implications and Conclusion

In this paper, we provided a joint analysis of production and sales decisions for a supply constrained new product diffusion process. Our diffusion model incorporated word of mouth effects of waiting (lost) customers in addition to adopters and innovators when the firm is supply constrained. We showed how the deeper analysis of the diffusion process characteristics could be used to prove the optimality for a formidable optimal control problem. Our analytical results showed that the presence of the scarcity effects arising from the product shortages affected the optimal sales plan, and hence optimal profits. The numerical experiments suggested that optimal capacity levels depended upon the scarcity effects. Hence, managers should anticipate them and include them while building sales and production plans.

Under the lost sales setting, though Kumar and Swaminathan (2003) suggested that the build up plan was always optimal, in contrast, we showed that the myopic policy could be optimal under the hype effect. Furthermore, the optimal build up varied with the magnitude of the retarding effects. Using an envelope theorem, we attempted to categorize the situations that favored the build up (myopic) policies respectively. Most of the prior work has not included word of mouth effects of waiting customers while forecasting the demand for the product. Our results analytically supported Simon and Sebastian (1987) and Jain et al. (1991) suggestions of including these effects during the forecasting process. Given that a little research has been carried out in the field of integrated operations and marketing decisions under product shortages, our results significantly add to the existing literature and also can be extended in several directions.

Limitations and Future Scope.

In the problem P1, we assume that the firm completely backorders the unmet demand. Note that the unmet demand {A (t) in the equation 1} and backorders at time t {A (t) in equation 2} are not the same in a partial lost sales situation. Hence, this paper does not explicitly address this situation. However, we can approximate it by explicitly adding the lost sales costs while calculating the per unit backorder cost (w). For example, one can use the idea suggested by Fisher et al. (2001) to calculate w.

Several models have been proposed to trace the optimal advertising path for new products (Mahajan et al. 2000). The objective of the advertising effort is to inform customers about a new product and encourage them to buy sooner (Krishnan and Jain 2006). Some of this research suggests that a firm should follow either an increasing or an increasing-decreasing advertising plan. However, these models assume that the firm has an infinite capacity, and thus, never incurs backorders (lost sales). However, under a supply constraint, the increase in demand caused by such policies sometimes can increase backorders (lost sales), and hence, in such situations, the above advertising policies may not remain optimal. E.g. Nintendo may curb the advertising for the console because of the shortage problems (The Escapist 2007). Hence, we believe that there is a need to investigate the behavior of advertising policies under supply constraints.

Preproduction can be used as a substitute for the capacity and thus serve as less costly mechanism to meet the sales (Ho et al. 2002). In this paper, we assume that the firm does not have any inventory at time zero. Also, one can divide the innovations into different segments according to the p/q ratio which also influences the diffusion dynamics. It will be worthwhile to investigate the benefits of preproduction in the presence of scarcity effects under different innovation segments.

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Appendix

Proof of Lemma 1.

For an optimal policy, because $0 \le S^*(t) \le m$ *for all* $t \ge 0$

$$p(m - Z^*(t)) \le z^*(t) \le (p + q_1 + q_2)(m - Z^*(t)) \quad \textit{for all } t \ge 0 \tag{24}$$

So, under an optimal policy, the cumulative demand is bounded by the following bounds

$$m(1-e^{-pt}) \le Z^*(t) \le m(1-e^{-(p+q_1+q_2)t}) \qquad (25)$$

The left hand side inequality of the equation (25) implies that

$$Z^*(t) \ge m - \frac{c}{(p+q_1+q_2)} \quad \textit{for all } t \ge t_c$$

Substituting the above inequality in the right hand side inequality of the equation (24) yields

$$z^*(t) \le c \quad \textit{for all } t \ge 0$$

Proof of Lemma 2.

At t_c^b, the growth rate is either positive or negative i.e. $z(t_c^b) \ge 0$ *or* $z(t_c^b) < 0$ under MP.

Case 1: When $z(t_c^b) \ge 0$ under MP $\Rightarrow$ the product is in the growth phase under MP at t_c^b . Because m is constant and $c \le z_p^m$, *using Lemma 1, we prove the result:* $t_c^m > t_c^b$.

Case 2: $z(t_c^b) < 0$ *under MP*: Case 2A: $z(t_c^b)$ *under SSD* $< z(t_c^b)$ under MP . Let t_1 and z_1 denote the time and the instantaneous demand when the SSD diffusion curve during the decline phase meets MP diffusion curve. By definition, $z(t_1)$ *under SSD* $< z(t_1)$ *under MP*. Now because $z_1 >$ c and because $z(t_c^b)$ *under SSD* $< z(t_c^b)$ under MP $\Rightarrow$ $t_c^m > t_c^b$

Case 2B: $z(t_c^b)$ *under SSD* $> z(t_c^b)$ under MP We rewrite the equation 12 at time t_c^b under SSD and MP respectively

$$z(t_c^b) \textit{ under SSD} = -pc + \frac{q_1}{m}\left\{-ct_c^b c + (m - Z(t_c^b)c\right\} + \frac{q_2}{m}\left\{-(A_u(t_c^b) + A_s(t_c^b))c\right\} \qquad (26)$$

$$z(t_c^b)\ under\ MP = -\ pz(t_c^b) + \frac{q_1}{m}\left\{-ct_c^b\ z(t_c^b) + (m - Z(t_c^b)c\right\} + \frac{q_2}{m}\left\{-(A_u(t_c^b) + A_s(t_c^b))z(t_c^b) + (m - Z(t_c^b)(a_u(t_c^b) + a_s(t_c^b))\right\} \quad (27)$$

Also, Z(t_c^b) under SSD > Z(t_c^b) under MP. Therefore, the comparison of the equations 26 and 27 $\Rightarrow z(t_c^b) > c\ under\ MP, then\ Lemma 1 \Rightarrow t_c^m > t_c^b$.

We prove the remaining result via contradiction. We assume that θ under SSD = θ under MP $\Rightarrow z(t) at\ t_c^b\ under\ SSD = z(t) at\ t_c^m\ under\ MP$. We again rearrange the equation 12 at time t_c^b under SSD and at time t_c^m under MP respectively

$$z(t_c^b)\ under\ SSD = -\ pc + \frac{q_1}{m}\left\{-ct_c^b c + (m - Z(t_c^b)c\right\} + \frac{q_2}{m}\left\{-(A_u(t_c^b) + A_s(t_c^b))c\right\} \quad (28)$$

$$z(t_c^m)\ under\ MP = -\ pc + \frac{q_1}{m}\left\{-ct_c^m c + (m - Z(t_c^m)c\right\} + \frac{q_2}{m}\left\{-(A_u(t_c^m) + A_s(t_c^m))c\right\} \quad (29)$$

The new assumption also implies that $Z(t_c^b) = Z(t_c^m)$. Because $t_c^m > t_c^b$, $A_u(t_c^b) + A_s(t_c^b) > A_u(t_c^m) + A_s(t_c^m) \Rightarrow z(t) at\ t_c^b\ under\ SSD < z(t) at\ t_c^m\ under\ MP$.

Also, z(t) = $\int z(t)dt$. Because z (t) is a smooth, continuous and decreasing function after time t_c, $z(t) under\ SSD < z(t) under\ MP$ for all $t > t_c$. That means that θ under SSD < θ under MP

This contradicts the result and thus proves the result that θ under SSD < θ under MP.

Proof of Proposition 1.

We prove this proposition using Lemma 2. Results of the Lemma 2 show that for the given c, because $t_c^m > t_c^b$ and θ *under SSD* < θ *under MP*, *backorder* costs are always increasing in $a_s(t)$ for all $t \geq 0$. Also, inventory holding cost, which depends upon i (t) is increasing in

$a_s(t)$ for all $t \geq 0$ (equation 6). Furthermore, positive unit margin obtained from selling a product unit is never increasing in time t (the contribution of the first two terms in the equation 2) as both price and the production cost does not change with $a_s(t)$ for all $t \geq 0$. Thus, the objective function (equation 2) is decreasing in $a_s(t)$ for all $t \geq 0$, *the* λ can be maximized by setting $a_s(t) = 0$ for all $t \geq 0$. Hence, for the general problem (P1), from equation 11, the myopic sales plan is always an optimal sales plan. Now, we use the restrictions imposed at optimality to preserve the feasibility to establish the optimal sales structure (equation 19). The feasibility of equations 6 and 10 imply that $s_t^* = x_t^*$ when $A_t^* > 0$ *and* $i_t^* = 0$. Also, equations 7 and 10 mean that $s_t^* = z_t^*$ when $A_t^* = 0$ *and* $i_t^* > 0$. And equations 6, 7, and 10 together imply that $s_t^* = \min(x_t^*, z_t^*)$ when $A_t^* = 0$ *and* $i_t^* = 0$. This proves the result.

Proof of Proposition 2.

Using the arguments similar to the Lemma 2, we can easily prove that $t_c^m < t_c^b$ when $q_2 < q_1$. As shown in LEMMA 2, via contradiction, we can easily prove that that θ under SSD > θ under MP when $q_2 < q_1$.

In the last section, we have shown that the objective function is the same in both regimes that is to minimize:

$$\lambda = \int_0^{t_c} z(t)dt - c\,t_c \Rightarrow m - \theta = \int_{t_c}^{\infty} z(t)dt - ct_c$$

$$\frac{\partial \lambda}{\partial \theta} < 0 \ \textit{and} \ \frac{\partial \lambda}{\partial t_c} < 0. \text{ Also, } \frac{\partial \theta}{\partial a_s(t)} > 0 \ \textit{and} \ \frac{\partial t_c}{\partial a_s(t)} > 0 \ \textit{for all } t \geq 0$$

Therefore using chain rule, $\dfrac{\partial \lambda}{\partial a_s(t)} < 0 \quad \textit{for all } t \geq 0$

Thus, the lost sales are always decreasing in the strategically delayed sales, and therefore, from equation 11, SSD is a better policy compared to MP when $q_2 < q_1$. Now, we compute the optimal amount of the strategically delayed sales that the firm should incur. Note that $a_s(t) \le z(t)$. Hence, strategically delayed sales can occur over a time period. Kumar and Swaminathan (2003) have shown that it is optimal to add the strategically delayed sales as early as possible. We can write $\lambda = A_u(T) + A_s(T)$. Because λ is decreasing in $a_s(t)$ while $A_s(T)$ is increasing in $a_s(t)$, $A_u(T)$ is decreasing in $a_s(t)$ for all $t \ge 0$. Hence, there exists a time t_s when $A_s(t_s) = A_u(T) - A_u(t_s)$. This is the maximum $A_s^*(t_s^*)$ that the firm can incur and yet can reduce the total lost sales. We have defined the build up policy (BU) as a SSD which strategically looses all the sales between 0 and t_s^*. As a result, when $z^*(t_s^*) > 0$, BU plan is always optimal. Note that when $q_2 \ge 0$, then $z^*(t_s^*) > 0$. When $z^*(t_s^*) \le 0$, because $\frac{\partial \lambda}{\partial a_s(t)} < 0$, it is always optimal for the firm to loose the sales between time 0 and t^* such that $z^*(t^*) = \varepsilon$ where ε is a very small positive number to reduce the total lost sales.

Proof of Theorem 1.

As shown in Klein (1998), we denote our state vector k(t) as {Z(t),S(t), i(t), A(t), n(t), a(t)} and control vector c(t) as {x(t), s(t)} and the profit function as π{c(t),k(t)} to represent P1 as [1]

$$\text{Max} \int_o^T e^{-rt} \pi(c(t), k(t) dt \ \ subject\ to\ \ k(t) = f(c(t), k(t)) \quad (30)$$

[1] In the equation 33, we choose T such that the market potential is almost exhausted and hence we can safely ignore the demand occurring after time T.

(Corresponding to constraints 3-7) with inequality constraints $g(c(t)) \geq 0$ (for constraints 8-9) and $k_3, k_4 \geq 0 (\textit{correspoding to equation } 10)$. Now, the Hamiltonian of the system is

$$H(c(t), k(t), \lambda(t)) = \pi(c(t), k(t)) + \lambda(t).f(c(t), k(t))$$

According to the maximum principle, the following inequality must hold at the optimal state vectors $k^*(t)$ and optimal Lagrange multiplier $\lambda^*(t)$.

$$H(c^*(t), k^*(t), \lambda^*(t)) \geq H(c(t), k^*(t), \lambda^*(t))$$

Note that the control vector enters linearly into the Hamiltonian for given

$k^*(t)$ and $\lambda^*(t)$. Therefore, the bang bang solution is optimal (Klein 1998). This implies that the sales are at the lower bound (zero) when the coefficient associated with $s^*(t)$ is negative. This results in situation where both $A_t^* > 0 \textit{ and } i_t^* > 0$. When it is positive, the sales plan operates according to the myopic policy (as shown in the Proposition 1) because only the myopic policy can sell at the maximum possible feasible upper bound. This proves the result.