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A NOTE ON THE ECONOMIC SIZING OF WAREHOUSES A LINEAR PROGRAMMING APPROACH

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ABSTRACT

In the past, researchers presented a linear programming formulation for the economic sizing of warehouses when demand is highly seasonal and public warehouse space is available on a monthly basis. The static model was extended for the dynamic sizing problem in which the warehouse size is allowed to change over time. By applying simplex routine, the optimal size of the warehouse to be constructed could be determined . In this paper, it is shown that for the static model, the optimal size could be determined more easily by calculating and comparing the costs associated with only a few values for the sizes of the private warehouse. The number of costs to be calculated is equal to the number of time periods in the planning horizon plus one. The dual of the dynamic model is shown to be a network problem.

1. INTRODUCTION

Ballou [1] offered a method for determining the most economical combination of private warehouse size and the public warehouse space. Hung and Fisk [3] gave an alternative formulation for two types of sizing problems - static and dynamic. They obtained solutions for a sample problem for both types of situations using UNIVAC FMPS. In this note, it is shown that the static problem could be solved easily without using any standard linear programming routines. The dynamic problem is shown to be the dual of a network problem.

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2. REVIEW

Ballou's [1] formulation of the problem is given below:
Suppose the planning horizon consists of T periods. It is assumed that the location for private warehouse is already determined. Any amount of public warehouse space can be leased in any month t. For each period t in the planning horizon, demands for the warehouse space are estimated. Since future demands cannot be known with certainty, the estimates are made according to pessimistic, most likely, and optimistic projections. There can be more or fewer estimates for each period. In general, it is assumed that there are n estimates, and for each estimate the probability of occurence is P_j , $j = 1,2,...,T$ and $\sum_j P_j = 1$.
Ballou showed that warehousing cost for period t can be computed from the following formula:

$$C_{tj} = C_0 X + C_v Y_{tj} + C_p (D_{tj} - Y_{tj}) \qquad (1)$$

where

C_{tj} = warehousing cost in period t under demand estimate schedule j;

C_0 = overheads and amortised capital expenditure per sq.ft per period;

X = size of private warehouse, in sq.ft;

C_v = variable private warehousing cost, per sq.ft. of storage per period;

C_p = variable public warehousing cost,per ft^2 of storage per period;

Y_{tj} = amount of private warehouse space used in period t, under estimate j;

D_{tj} = demand for storage space, in ft^2 in period t, under estimate j.

It is also assumed that only a fraction f of the private warehouse space can be used for storage, so that:

$$Y_{tj} = fX \quad \text{if } D_{tj} > fX \qquad (2)$$
$$= D_{tj} \quad \text{if } D_{tj} <= fX$$

The total expected cost for the planning horizon is:

$$EC = \sum_{t=1}^{T}\sum_{j=1}^{n} P_j C_{tj} \qquad (3)$$

Thus the problem of sizing a private warehouse is to determine the warehouse size X and the allocation of storage, Y_{tj}'s such that EC is minimized.

A simple alternative to Ballou's method of finding optimal warehouse size was given by Hung and Fisk [2]. They used linear programming formulation. They first replace, for each demand period t, the set of demand estimates and their corresponding probabilities of occurrence with the expected value of demand D_t:

$$D_t = \sum_{j=1}^{n} P_j D_{tj}$$

Similarly, the amount of private warehouse space used in each period t under estimate j is replaced by Y_t, the expected value of warehouse space used in period t.

The linear programming formulation developed for the static problem is as follows:

$$(P): \quad Minimize\ EC = \sum_{t=1}^{T} [\ C_0 X + C_v Y_t + C_p (D_t - Y_t)\ ] \qquad (4)$$

subject to:

$$Y_t \le fX, \quad t=1,2,\ldots,T \qquad (5)$$

$$Y_t \le D_t \quad t=1,2,\ldots,T \qquad (6)$$

$$X \ge 0,\ Y_t \ge 0 \quad t=1,2,\ldots,T \qquad (7)$$

In this model, the amount of public warehouse space hired in month t is $(D_t - Y_t)$ which can vary from month to month.

3. ALTERNATIVE METHOD OF SOLUTION

By making the variable substitution S = f.X , we reformulate the problem as follows:

$$(P1): \quad Minimize\ EC = \sum_{t=1}^{T} [\ \frac{C_0}{f} S + C_v Y_t + C_p (D_t - Y_t)\ ] \qquad (8)$$

subject to:

$$Y_t \le S, \quad t=1,2,\ldots,T \qquad (9)$$

$$Y_t \le D_t \quad t=1,2,\ldots,T \qquad (10)$$

$$S \ge 0,\ Y_t \ge 0 \quad t=1,2,\ldots,T \qquad (11)$$

Each row of the constraints has at most two ones. If a row has two ones, then one of them is +1 and the other is -1. So, the dual has variables with at most two ones in each column. If a dual variable has two ones, then they are of opposite sign.

Remark: Dual of problem (P) is a network problem and hence can be solved efficiently. However, as shown below, the problem can be solved without applying network algorithm.

Assume that the optimal solution to problem (P) is

$$X = X^* \quad \text{and } Y_t = Y_t^* \quad \text{for } t = 1,2,\ldots,T$$

so that the optimal solution to problem (P1) is

$$S = S^* = f.X^* \text{ and } Y_t = Y_t^* \text{ for } t = 1,2,\ldots,T.$$

CLAIM 1:

If $C_p \le C_v$, then $S^* = 0$.

This follows from the structure of the objective function.

We shall assume that $C_p > C_v$.

CLAIM 2:

If $C_p \le C_v + \frac{C_0}{f}$ *, then there exists an optimal solution with* $S^*=0$

Proof:

Suppose, $C_p \le C_v + \frac{C_0}{f}$ *, and* S^* *is positive.*

Let $V_0 = \{ t \; ; \; 0 < D_t < S^* \}$

$V_1 = \{ t \; ; \; D_t \ge S^* \}$ and

$V_2 = \{ t \; ; \; D_t = 0 \}$

Then, $Y_t^* = D_t$ for $t \in V_0$

$= S^*$ for $t \in V_1$

$= 0$ for $t \in V_2$

Let $S = S^* - \Delta S^*$,

$Y_t = Y_t^*$ • for $t \in V_0$

$= Y_t^* - \Delta S^*$ for $t \in V_1$ and

$= Y_t^* = 0$ for $t \in V_2$

where ΔS^* is a small positive quantity.

The values of S and Y_t satisfy constraints (9),(10) and (11).

Let T_0, T_1 and T_2 be the cardinalities of V_0, V_1 and V_2 respectively.

The objective function value will be greater than or equal to the optimal value i.e.

$$\sum_{t=1}^{T} \frac{C_0}{f}(S^*-\Delta S^*) + \sum_{t\in V_0} C_v Y_t^* + \sum_{t\in V_1} C_v(Y_t^*-\Delta S^*) + \sum_{t\in V_1} C_p(D_t - Y_t^* + \Delta S^*) -$$

$$\sum_{t=1}^{T} \frac{C_0}{f} S^* - \sum_{t\in V_0} C_v Y_t^* - \sum_{t\in V_1} C_v Y_t^* - \sum_{t\in V_1} C_p(D_t - Y_t^*) \geq 0$$

After simplification, this reduces to

$$\Delta S^* \left(\frac{C_0}{f} T + C_v T_1 - C_p T_1\right) \leq 0$$

$$i.e. \qquad C_p T_1 \geq \frac{C_0}{f} T + C_v T_1$$

$$or \qquad C_p \geq \frac{C_0}{f}\,\frac{T}{T_1} + C_v$$

But, since

$$C_p \leq C_v + \frac{C_0}{f} \quad ,$$

it follows that

$$C_p = \frac{C_0}{f} + C_v \quad and \quad T = T_1$$

This implies $T_0 = T_2 = 0$.

Hence

$$S^* \leq Min_t\, D_t$$

Now, set $\Delta S^* = S^*$ and we get another optimal solution with optimal value of $S = 0$ and $Y_t = 0$.

LEMMA : There exists an optimal solution to problem (P_1) such that either $S = 0$ or $S = D_t$ for some t, $t = 1,2,\ldots,T$.

Proof:

If $C_p \leq C_v + \frac{C_0}{f}$, *by Claim 2 , the lemma follows.*

Suppose $C_p > C_v + \frac{C_0}{f}$.

Assume that the optimal solution to the problem is:

$S = S^*$ and $Y_t = Y_t^*$ for $t = 1,2,\ldots,T$.

Suppose that $S^* > 0$ and not equal to D_t for any t, $t = 1,2,\ldots,T$.

Define sets U_1 and U_2 such that

$$S^* < D_t\,, \quad for\ t \in U_1$$

$$S^* > D_t\,, \quad for\ t \in U_2$$

Note that U_1 and U_2 cover the entire planning horizon.

Let T_1 and T_2 be the cardinalities of U_1 and U_2 respectively.

Since $C_p > C_v + \frac{C_0}{f}$, *it can be seen that*

for any t in U_1, $Y_t^* = S^*$ and

for any t in U_2, $Y_t^* = D_t$

so that the optimal cost is

$$\frac{C_0}{f} TS^* + C_v \sum_{t \in U_1} S^* + C_p \sum_{t \in U_1} (D_t - S^*) + C_v \sum_{t \in U_2} D_t$$

After simplification, this cost is

$$\frac{C_0}{f} TS^* + (C_v - C_p) S^* T_1 + C_p \sum_{t \in U_1} D_t + C_v \sum_{t \in U_2} D_t \qquad (12)$$

Let

$S = S^* + \Delta S^*$ *where* ΔS^* *is a small change in the value of* S^*.

Since $S^* > 0$ and not equal to D_t for any t, t = 1,2,...,T, it follows that for ΔS^* sufficiently small, ($S^* + \Delta S^*$) > 0 and not equal to D_t for any t, t = 1,2,...,T. Now, set

$$Y_t = (S^* + \Delta S^*) \quad \textit{for } t \in U_1$$

and

$$Y_t = D_t \quad \textit{for } t \in U_2$$

The values of S and Y_t satisfy the constraints (9), (10) and (11). The value of the objective function will now be:

$$\frac{C_0}{f} T(S^* + \Delta S^*) + C_v \sum_{t \in U_1} (S^* + \Delta S^*) + c_v \sum_{t \in U_2} D_t + C_p \sum_{t \in U_1} [D_t - (S^* + \Delta S^*)]$$

This simplifies to:

$$\frac{C_0}{f} TS^* + (C_v - C_p) S^* T_1 + C_p \sum_{t \in U_1} D_t + C_v \sum_{t \in U_2} D_t + [\frac{C_0}{f} T + (C_v - C_p) T_1] \Delta S^* \qquad (13)$$

Now (13) - (12) gives

$$[\frac{C_0}{f} T + (C_v - C_p) T_1] \Delta S^* \qquad (14)$$

If $\frac{C_0}{f} T + (C_v - C_p) T_1$ *is not equal to* 0, *choose* ΔS^*

as either positive or negative so that (14) is negative. This contradicts the optimality of the solution $S = S^*$ and

$Y_t = Y_t^*$ for $t = 1,2,\ldots,T$.

Hence

$$\frac{C_0}{f} T + (C_v - C_p) T_1 = 0 .$$

Since

$$C_p > C_v + \frac{C_0}{f} ,$$

it follows that $T_1 < T$ and $T_2 > 0$.

Now choose

$$\Delta S^* = -\min_{t \in U_2} [S^* - D_t]$$

Then, $S^* + \Delta S^*$ equals D_t for some $t \in U_2$ and we have an alternate optimal solution with $S = S^* + \Delta S^*$. This completes the proof of the lemma.

Method of Solution

The optimal solution can be obtained by enumeration of (T+1) possible values for S. These are $S = 0$ and $S = D_t$ for $t = 1,2,3,\ldots,T$. For each possible value, the corresponding values of Y_t , $t = 1,2,\ldots,T$ and the associated cost can be calculated. A value of S that gives the least cost is an optimal solution.

Dynamic Warehouse Sizing Problem

The dynamic warehouse sizing problem as formulated by Hung and Fisk is as follows:

$$Min \sum_{t=1} [C_0 X_t + C_e^t W_t + C_r^t Z_t + C_v Y_t + C_p (D_t - Y_t)]$$

subject to

$$Y_t - f X_t \leq 0 , \qquad t = 1,2,\ldots,T.$$

$$Y_t \leq D_t , \qquad t = 1,2,\ldots,T.$$

$$X_t - X_{t-1} - W_t + Z_t = 0 , \qquad t = 1,2,\ldots,T.$$

$$X_t, Y_t, W_t, Z_t \geq 0 , \qquad t = 1,2,\ldots,T.$$

where

X_t = warehouse size in period t, X_0 given

$(X_t - X_{t-1})^+ = X_t - X_{t-1}$ if $X_t \geq X_{t-1}$

$= 0$ otherwise

$(X_t - X_{t-1})^- = X_{t-1} - X_t$ if $X_t \leq X_{t-1}$

$= 0$ otherwise

$W_t = (X_t - X_{t-1})^+$ = amount of expansion in period t

$Z_t = (X_t - X_{t-1})^-$ = amount of reduction in period t

C_e^t = per unit expansion cost in period t

C_r^t = per unit reduction cost in period t.

The definition of other variables and costs remain the same.

Let $S_t = f\ X_t$, $U_t = f\ W_t$ and $V_t = f\ Z_t$ for $t = 1,2,\dots,T$.

Now the problem becomes

$$Min \ \frac{1}{f}\sum_{t=1}^{T} [C_0 S_t + C_e^t U_t + C_r^t V_t + f(C_v - C_p) Y_t]$$

subject to

$$-Y_t + S_t \geq 0 \quad , \quad t = 1,2,\dots,T \qquad (15)$$

$$-Y_t \geq -D_t \quad , \quad t = 1,2,\dots,T \qquad (16)$$

$$S_t - S_{t-1} - U_t + V_t = 0 \quad , \quad t = 1,2,\dots,T \qquad (17)$$

$$S_t,\ Y_t,\ U_t,\ V_t \geq 0 \quad , \quad t = 1,2,\dots,T \qquad (18)$$

where $S_0 = f\ X_0$ which is given.

This is equivalent to

$$Min \ \sum_{t=1}^{T} [C_0 S_t + C_e^t U_t + C_r^t V_t + f(C_v - C_p) Y_t]$$

subject to the same constraints.

Let α_t, β_t and γ_t, $t = 1,2,\dots,T$ be the dual variables associated with constraints (15),(16) and (17) respectively.

Now, the dual problem is

$$Max \ [S_0\gamma_1 - \sum_{t=1}^{T} D_t\beta_t]$$

subject to

$$-\alpha_t - \beta_t \leq f\ (C_v - C_p) \quad , \quad t = 1,2,\dots,T$$

$$\alpha_t + \gamma_t - \gamma_{t+1} \leq C_0 \quad , \quad t = 1,2,\dots,T$$

$$-\gamma_t \leq C_e^t \quad , \quad t = 1,2,\dots,T$$

$$\gamma_t \leq C_r^t \quad , \quad t = 1,2,\dots,T$$

$\alpha_t \geq 0$, $\beta_t \geq 0$, γ_t unrestricted in sign, $t = 1,2,...,T$

The problem is equivalent to

$$Max \ [\sum_{t=1}^{T} -D_t\beta_t] + S_0\gamma_1$$

subject to

$$- \alpha_t - \beta_t \leq f \ (C_v - C_p) \ , \qquad t = 1,2,...,T$$

$$\alpha_t + \gamma_t - \gamma_{t+1} \leq C_0 \qquad t = 1,2,...,T$$

$$- C_e^t \leq \gamma_t \leq C_r^t \qquad , \qquad t = 1,2,...,T$$

$$\alpha_t \geq 0, \ \beta_t \geq 0, \qquad , \qquad t = 1,2,...,T$$

The dual formulation is a bounded network flow problem with lower and upper limits on the variables γ_t , $t = 1,2,...,T$.

REFERENCES

1. Ronald H. Ballou, Business Logistics Management, Prentice Hall, Englewood Cliffs, 1974.

2. Donald J. Bowersox, Logisistical management, Macmillan, 1974.

3. M.S. Hung and John C. Fisk, Economic Sizing of Warehouses - A Linear Programming Approach,Computers and Operations Research, Vol 11, No. 1, 1984.