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ECONOMIC SIZING OF WAREHOUSES SOME EXTENSIONS

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ECONOMIC SIZING OF WAREHOUSES - SOME EXTENSIONS

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ABSTRACT

In the past, researchers presented a linear programming formulation for the economic sizing of warehouses when demand is highly seasonal and public warehouse space is available on a monthly basis. The static model was extended for the dynamic sizing problem in which the warehouse size is allowed to change over time. By applying simplex routine, the optimal size of the warehouse to be constructed could be determined . In this paper, an alternative and simple method of arriving at an optimal solution for the static problem is given. Four extensions of the static model are given. These extensions involve initial warehouse capacity,costs varying over time, economies of scale in capital expenditure and/or operating cost and stochastic version. The dynamic warehouse sizing problem is shown to be a network flow problem which could be solved by using network flow algorithms. The structure of an optimal solution is also given. The concave cost version of the dynamic warehouse sizing problem is also discussed.

1. INTRODUCTION

In an earlier paper [5], it is shown that the static warehouse sizing problem given by Hung and Fisk [3] can be solved easily without using any standard linear programming routines. The solution method consists of enumerating the costs corresponding to (T+1) possible values of the warehouse size. In this paper, the nature of the optimal solution is studied and a straight forward method of arriving at the optimal size of private warehouse is given. Four extensions of the static case are considered. These

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consist of initial warehouse capacity, costs varying with time, economies of scale in capital expenditure and/or operating cost and stochastic version. Methods of arriving at the optimal solution are discussed. The dynamic warehouse sizing problem is shown to be a network flow problem which could be solved by using network flow algorithms. The structure of an optimal solution is derived. This leads to an efficient dynamic programming algorithm for the concave cost version of the dynamic warehouse sizing problem.

2. REVIEW AND NOTATION

Suppose the planning horizon consists of T periods. It is assumed that the location for private warehouse is already determined. Any amount of public warehouse space can be leased in any month t. For each period t in the planning horizon, demands for the warehouse space are estimated. In general, it is assumed that there are n estimates, and for each estimate the probability of occurrence is P_j , $j = 1,2,...,n$ and $\sum_j P_j = 1$.

Ballou [1] showed that warehousing cost for period t can be computed from the following formula:

$$C_{tj} = C_0 X + C_v Y_{tj} + C_p (D_{tj} - Y_{tj})$$

where

C_{tj} = warehousing cost in period t under demand estimate schedule j;

C_0 = overheads and amortised capital expenditure per sq.ft per period;

X = size of private warehouse, in sq.ft;

C_v = variable private warehousing cost, per sq.ft. of storage per period;

C_p = variable public warehousing cost, per ft^2 of storage per period;

Y_{tj} = amount of private warehouse space used in period t, under estimate j;

D_{tj} = demand for storage space, in ft^2 in period t, under estimate j.

It is also assumed that only a fraction f of the private warehouse space can be used for storage, so that:

$$Y_{tj} = \begin{cases} f\,X & \text{if } D_{tj} > f\,X \\ D_{tj} & \text{if } D_{tj} \leq fX \end{cases}$$

The total expected cost for the planning horizon is:

$$EC = \sum_{t=1}^{T} \sum_{j=1}^{n} P_j \, C_{tj}$$

Thus the problem of sizing a private warehouse is to determine the warehouse size X and the allocation of storage, Y_{tj}'s such that EC is minimized.

A simple alternative to Ballou's method of finding optimal warehouse size was given by Hung and Fisk [3]. They used linear programming formulation. They first replace, for each period t, the set of demand estimates and their corresponding probabilities of occurrence with the expected value of demand D_t:

$$D_t = \sum_{j=1}^{n} P_j \, D_{tj}$$

Similarly, the amount of private warehouse space used in each period t under estimate j is replaced by Y_t, the expected value of warehouse space used in period t.

The linear programming formulation developed for the static problem is as follows:

$$(P) : \quad \text{Minimize } EC = \sum_{t=1}^{T} [\, C_0 X + C_v Y_t + C_p (D_t - Y_t) \,]$$

subject to :

$$Y_t \le f X , \qquad t = 1,2,...,T$$

$$Y_t \le D_t , \qquad t = 1,2,...,T$$

$$X \ge 0 , \; Y_t \ge 0 , \qquad t = 1,2,...,T$$

In this model, the amount of public warehouse space hired in period t is $(D_t - Y_t)$ which can vary from period to period.

A simple method of arriving at an optimal solution without recourse to linear programming is given in [5]. The method consists of enumerating the total cost coresponding to (T+1) possible values of the warehouse size.

3. ALTERNATIVE METHOD OF SOLUTION

We first make the variable substitution $S = f.X$ and denote C_0 / f by C_f and $T\, C_f$ by T_f. Omitting the constant term $\sum_{t=1}^{T} C_p D_t$ we reformulate the problem as follows:

$$(P1) : Z^* = \text{Min } EC = T_f\, S + \sum_{t=1}^{T} (C_v - C_p)\, Y_t \qquad (1)$$

subject to:

$$Y_t \le S , \qquad t = 1,2,...,T \qquad (2)$$

$$Y_t \le D_t , \qquad t = 1,2,...,T \qquad (3)$$

$$S \ge 0 , \; Y_t \ge 0 , \qquad t = 1,2,...,T \qquad (4)$$

If any $D_t = 0$, then the corresponding $Y_t = 0$ and it can be dropped from the problem. So without loss of generality we assume that $D_t > 0$ for $t = 1,2,\ldots,T$.

Each row of the constraint set has at most two ones. If a row has two ones, then one of them is +1 and the other is -1. So, the dual problem has variables with at most two ones in each column. If a dual variable has two ones, then they are of opposite sign.

Remark 1 : The dual of problem (P_1) is a network problem and hence can be solved efficiently. However, as shown below, the problem can be solved without applying network algorithm.

Let S^* , Y_t^* , $t = 1,2,\ldots,T$ be an optimal solution to (P1).

Remark 2 : $Z^* \leq 0$. If $C_p \leq C_v$, then there exists an optimal solution with $S^* = 0$ and $Z^* = 0$.

This follows from the structure of the objective function and the constraints.

We henceforth assume that $C_p > C_v$.

Remark 3 : If $C_p \leq C_v + C_f$, then there exists an optimal solution with $S^* = 0$.

Proof:

Suppose $C_p \leq C_v + C_f$.

Now the objective function (1) is such that

$$T\,C_f\,S + \sum_{t=1}^{T} (C_v - C_p)\,Y_t \geq T\,C_f\,S + \sum_{t=1}^{T} (C_v - C_p)\,S$$

$$= T\,S\,(C_v - C_p + C_f) \geq 0.$$

where the first inequality follows from $Y_t \leq S$, $t=1,2,\ldots,T$ and

$C_v - C_p < 0$ while the last inequality follows from $C_v - C_p + C_f \geq 0$. Consequently $S = 0$ and $Y_t = 0$, $t = 1,2,\ldots,T$ is an optimal solution to (P1). ∎

We henceforth assume that $C_p > C_v + C_f$. We now show that an optimal solution to (P1) can be obtained very easily without recourse to linear programming.

We first sort the demands D_t in increasing order with ties broken arbitrarily.

Let $D_{[i]}$ denote the demand in the i th sorted position where $D_{[i]} \leq D_{[i+1]}$ for $i = 1,2,\ldots,T-1$.

Let $u = \lfloor T_f/(C_p - C_v) \rfloor$ denote the largest integer less than or equal to $T_f / (C_p - C_v)$. Note that $0 \leq u \leq T-1$ since $C_p - C_v > C_f$.

Let $k = T - u$.

<u>Theorem 1</u> : There exists an optimal solution to (P1) such that $S = D_{[k]}$. Moreover, $S = D_{[k]}$ is a unique optimal solution if

i) $D_{[k]} = D_{[k+1]}$ or

ii) $u = \lfloor T_f / (C_p - C_v) \rfloor < T_f / (C_p - C_v)$.

Proof: Let S^* and Y_t^* , $t = 1,2,\ldots,T$ be an optimal solution such that $S^* \neq D_{[k]}$. Let Z^* be the corresponding objective function value.

We first show that $S^* \geq D_{[k]}$.

<u>Claim 1</u> : $S^* \geq D_{[k]}$.

Proof : Suppose $S^* < D_{[k]}$.

Let $R = \{ t \mid D_t > S^* \}$ and $r = | R |$.

Note that $r \geq u+1 > T_f / (C_p - C_v)$.

Let $\Delta S = \min_{t \in R} \{ D_t - S^* \} > 0.$

Consider the following solution to (P1) which is clearly feasible:

$$\overline{S} = S^* + \Delta S$$

$$\overline{Y}_t = Y_t^* + \Delta S \quad \text{for } t \in R$$

$$= Y_t^* \quad \text{otherwise.}$$

Let $\overline{Z}$ be the corresponding objective function value.

$$\text{Now } \overline{Z} - Z^* = T_f \, \Delta S + \sum_{t \in R} (C_v - C_p) \, \Delta S$$

$$= \Delta S \, [T_f + r \, (C_v - C_p)]$$

$$< \Delta S \, [T_f + (T_f/(C_p - C_v)) \quad (C_v - C_p)]$$

$$= 0$$

where the strict inequality follows from

$|R| = r \geq u+1 > T_f \, / \, (C_p - C_v)$ and $(C_v - C_p) < 0$.

This implies that $\overline{Z} < Z^*$ which is a contradiction. Hence $S^* \geq D_{[k]}$.

Suppose now that $S^* > D_{[k]}$.

Let $W = \{ t \mid D_t < S^* \}$ and $w = |W|$. Note that $w \geq k = T-u$.

Let $\Delta S = \min_{t \in W} \{ S^* - D_t \} > 0$.

Consider the following solution to (P1) which is clearly feasible:

$$\overline{S} = S^* - \Delta S$$

$$\overline{Y}_t = Y_t^* \quad \text{for } t \in W$$

$$= Y_t^* - \Delta S \quad \text{otherwise}$$

Let $\overline{Z}$ be the corresponding objective function value.

$$\text{Now } Z^* - \overline{Z} = T_f \, \Delta S + \sum_{t \notin W} (C_v - C_p) \, \Delta S$$

$$\geq \Delta S \, [T_f + u \, (C_v - C_p)]$$

$$\geq \Delta S \, [T_f + (T_f/(C_p - C_v)) \, (C_v - C_p)] = 0 \qquad (5)$$

where the first inequality follows from $T - |W| = T - w \leq u$ and

$(C_v - C_p) < 0$ while the second inequality follows from $u \leq T_f / (C_p - C_v)$ and $(C_p - C_v) < 0$. This implies that $Z^* \geq \overline{Z}$. If $Z^* > \overline{Z}$, we have a contradiction.

Suppose $Z^* = \overline{Z}$. This implies that $\overline{S}$, $\overline{Y}_t$, $t = 1,2,...,T$ is an alternate optimal solution to (P1). Furthermore, $\overline{Z} = Z^*$ if and only if equality holds throughout (5), i.e. if and only if $T - w = u = T_f/(C_p - C_v)$. Then $k = T - u = w$.

Since $S^* > D_{[k]}$ and $k = w$, it follows that $D_{[k]} < S^* \leq D_{[k+1]}$ and $\overline{S} = S^* - \Delta S = D_{[k]}$. Consequently, $S = D_{[k]}$ is an alternate optimal solution to (P1) and the first part of the theorem follows.

Now, as shown above, $S = S^* > D_{[k]}$ is optimal to (P1), only if $S^* \leq D_{[k+1]}$ and $u = T_f/ (C_p - C_v)$. But this is impossible if $D_{[k]} = D_{[k+1]}$ or $u < T_f /(C_p - C_v)$. Thus the theorem follows. ■

<u>Remark 4</u> : If $D_{[k+1]} > D_{[k]}$ and $u = T_f / (C_p - C_v)$, then there exists an optimal solution for all values of S such that $D_{[k]} \leq S \leq D_{[k+1]}$.

Let D^j, $j = 1,2,...,q$ denote the q distinct values of the demands $D_{[i]}$ in increasing order.

Let $W_j = \{ t \mid D_t > D^j \}$ and $w_j = |W_j|$.

For each specified non-negative value of S, let

$$Z(S) = T_f S + \text{Min} \sum_{t=1}^{T} (C_v - C_p) Y_t$$

$$\text{subject to:} \quad Y_t \leq S$$

$$Y_t \leq D_t , \qquad t=1,2,...,T$$

Let $Z'(S)$ denote the derivative of $Z(S)$ at the values of S at which it is differentiable.

Lemma 1 : $Z(S)$ is a piecewise linear continuous function with $Z(0) = 0$ and the break points at D^j, $j = 1,2,\ldots,q$. The derivative of $Z(S)$ is given by

$$Z'(S) = T_f + (C_v - C_p)\,T \qquad \text{for } 0 < S < D^1$$

$$= T_f + (C_v - C_p)\,w_j \qquad \text{for } D^j < S < D^{j+1},\ j=1,2,\ldots,q-1$$

$$= T_f \qquad \text{for } D^q < S$$

Furthermore,

$$Z'(S) < 0 \qquad \text{for } S < D_{[k]} \text{ and } S \neq D^j \text{ for any } j$$

$$Z'(S) \geq 0 \qquad \text{for } S \geq D_{[k]} \text{ and } S \neq D^j \text{ for any } j$$

$$Z'(S) = 0 \qquad \text{for } D_{[k]} < S < D_{[k+1]} \text{ if } D_{[k]} \neq D_{[k+1]} \text{ and } u = T_f/\,(C_p - C_v).$$

Proof : The functional form and the derivative of $Z(S)$ follow from the relationships $C_v < C_p$, $Y_t = \text{Min}\ \{ S, D_t \}$ and the definition of $D_{[k]}$.

4. EXTENSIONS

We consider four extensions of the static model discussed in the previous sections. These extensions consist of

i) initial warehouse capacity

ii) costs varying over time

iii) economies of scale in capital expenditure and/or operating cost and

iv) stochastic version given by Ballou.

4.1 Initial Warehouse Capacity

Suppose there is an initial warehouse capacity of $X_0 > 0$.

Let $S_0 = f\ X_0$. Denoting U to be the additional private warehouse to be constructed and $S = U + S_0$, problem (P1) now becomes

$$(\text{P2}) \qquad \text{Minimize} \quad T_f\ S + \sum_{t=1}^{T} (\ C_v - C_p\ )\ Y_t - T_f\ S_0$$

$$\text{subject to} \qquad Y_t \le S \qquad t = 1,2,\ldots,T \qquad (6)$$

$$0 \le Y_t \le D_t \qquad t = 1,2,\ldots,T \qquad (7)$$

$$S \ge S_0 \qquad (8)$$

Let (P3) denote the problem (P2) without constraint (8). Now applying Theorem 1, an optimal solution

$S = S^*$, $Y_t = Y_t^*$, $t = 1,2,\ldots,T$, to (P3) can be found.

If $S^* \ge S_0$, clearly $U = S^* - S_0$, $Y_t = Y_t^*$, $t = 1,2,\ldots,T$ is an optimal solution to (P2). On the other hand, if $S^* < S_0$, it follows that (P2) has an optimal solution with $S = S_0$.

Consequently, if $S^* < S_0$, an optimal solution to (P2) is given by $S = S_0$, $Y_t = S_0$ for all t such that $D_t \ge S_0$ and $Y_t = D_t$ for all t such that $D_t < S_0$.

4.2 Costs varying over time

The variable costs C_v and C_p associated with private and public warehouses are now time dependent. Let C_{vt} and C_{pt} respectively represent the variable private and public warehouses cost in time period t .

Now the optimization problem is:

$$(P4) \qquad \text{Minimize } EC = Z = T_f\ S + \sum_{t=1}^{T} (C_{vt} - C_{pt})\ Y_t$$

subject to constraints (2), (3) and (4).

Note that if $C_{vt} \geq C_{pt}$ for any t, then the corresponding Y_t will be 0 in an optimal solution to (P4).

Let $V = \{ t \mid C_{vt} < C_{pt} \}$

The problem (P4) is equivalent to

$$Z^* = \text{Min} \quad T_f\ S + \sum_{t \varepsilon V} (C_{vt} - C_{pt})\ Y_t$$

subject to constraints (2), (3) and (4).

<u>Remark 5</u> : As in Section 3, $Z^* \leq 0$.

As in Section 3, we first sort the demands D_t, $t \varepsilon V$ in increasing order with ties broken arbitrarily.

Let D^j, $j = 1,2,\ldots,q$ denote the q distinct values of the sorted demands D_t, $t \varepsilon V$, i.e. $D^j < D^{j+1}$ for $j = 1,2,\ldots,q-1$.

Let $W_j = \{ t\varepsilon V \mid D_t \geq D^j \}$ and

$$C_j = \sum_{t\varepsilon W_j} (C_{pt} - C_{vt}) \quad ; \qquad j = 1,2,\ldots,q.$$

Note that $C_j > C_{j+1}$ for $j = 1,2,\ldots,q-1$.

We now have the following result:

<u>Theorem 2</u> : i) If $T_f \geq C_1$, then $S = 0$ is an optimal solution to (P4).

ii) If $T_f \leq C_q$, then $S = D^q$ is an optimal solution to (P4).

iii) Suppose $C_i \geq T_f \geq C_{i+1}$ for some i, $1 \leq i \leq q-1$, then $S = D^i$ is an optimal solution to (P4).

The proof of this theorem is similar to the proof of Theorem 1 and the details are omitted.

4.3 Economies of Scale

Suppose there are economies of scale in the overheads and amortized capital expenditure costs, i.e. this cost is a concave function, say $g(X)$, of the size of the private warehouse X. Making variable substitution $X = S/f$ and denoting $h(S) = g(S/f)$, it follows that $h(S)$ is a concave function of S for $S \geq 0$ since $f > 0$.

The problem now becomes

$$\text{(P5)} \qquad \text{Minimize } h(S) + \sum_{t=1}^{T} (C_v - C_p) Y_t$$

subject to constraints (2), (3) and (4).

Since the objective function is a concave function and the constraints are linear, it follows (see for instance Hadley [2]) that (P5) has an optimal solution which is an extreme point of the polyhedron defined by the constraints (2), (3) and (4). But a local optimal solution is not necessarily a global optimal solution and an enumeration of the extreme points is required to solve (P5). The lemma below characterizes the extreme points and thereby implies that (P5) can be solved easily.

<u>Lemma 2</u> : $S = 0$ or $S = D_t$ for some $t = 1,2,...,T$ in every extreme point of the polyhedron defined by the constraints (2), (3) and (4).

Proof : Let $S = S^* > 0$, $Y = Y_t^*$, $t = 1,2,...,T$ be an extreme point such that $S^* \neq D_t$ for any t. Suppose $Y_t^* > 0$ for some t.
Since $S^* \neq D_t$, it follows that the slack variable in one of the constraints $Y_t \leq S$ or $Y_t \leq D_t$ must be positive. On the other hand, suppose $Y_t^* = 0$ for some t. Then the slack variable in both of the above constraints must be positive since S and D_t are positive. Thus for every $t = 1,2,...,T$, there are at least two distinct variables, including possibly Y_t but not counting S, which are positive. Furthermore, since $S > 0$, we have at least 2T+1 variables which are positive. But every extreme point can have at most 2T variables positive, since there are only 2T constraints. Thus the lemma follows.

<u>Remark 6</u> : In order to solve (P5), we need to consider only T+1 values of S corresponding to 0 and D_t, $t = 1,2,...,T$.

4.4 Stochastic Case

The last extension we consider is the stochastic version given by Ballou [1]. The demand in each period is now not known with certainty but is specified by a discrete probability distribution. For $t = 1,2,...,T$, let D_{tj} , $j = 1,2,...n_t$ be the possible demands in period t with corresponding probabilities p_{tj} , $j = 1,2,...,n_t$.

Making the variable substitution S = f X and omitting the constant term $\sum_{t=1}^{T} \sum_{j=1}^{n_t} p_{tj} C_p D_{tj}$, the stochastic version of the problem can be formulated as a linear program as follows:

$$\text{(P6)} \qquad \text{Minimize } EC = T_f\ S + \sum_{t=1}^{T} \sum_{j=1}^{n_t} [\ p_{tj}\ (\ C_v - C_p\ )]\ Y_{tj}$$

$$\text{subject to:} \quad Y_{tj} \leq S, \qquad t = 1,2,\ldots,T;\ j = 1,2,\ldots,n_t$$

$$Y_{tj} \leq D_{tj}, \qquad t = 1,2,\ldots,T;\ j = 1,2,\ldots,n_t$$

$$S \geq 0,\ Y_{tj} \geq 0, \qquad t = 1,2,\ldots,T;\ j = 1,2,\ldots,n_t$$

Consider each combination of t and j as a separate time period. We have $\sum_{t=1}^{T} n_t$ time periods. Now p_{tj} ($C_v - C_p$) is the variable cost which depends upon the time period represented by the combination of t and j. Hence (P6) is identical to (P4).

An optimal solution to (P6) is easily obtained as in the case of (P4).

<u>Remark 7</u> : In order to solve the stochastic version, it is incorrect to consider the expected demand in period t and solve the associated problem (P1). For instance, the stochastic version as given by Ballou [1] has an optimal solution with X = 28,953. But if we take the expected demand and solve the associated problem (P1) an optimal solution is given by X = 26,577.

5. Dynamic Warehouse Sizing Problem

The dynamic warehouse sizing problem as formulated by Hung and Fisk [3] is as follows:

$$\text{Min} \sum_{t=1}^{T} [\, C_0 X_t + C_e^t W_t + C_r^t Z_t + C_v Y_t + C_p (D_t - Y_t)]$$

subject to

$$Y_t - f X_t \leq 0 , \qquad t = 1,2,...,T.$$

$$Y_t \leq D_t , \qquad t = 1,2,...,T.$$

$$X_t - X_{t-1} - W_t + Z_t = 0 , \qquad t = 1,2,...,T.$$

$$X_t, Y_t, W_t, Z_t \geq 0 , \qquad t = 1,2,...,T.$$

where

X_t = warehouse size in period t, X_0 is given.

W_t = amount of expansion in period t

Z_t = amount of reduction in period t

C_e^t = per unit expansion cost in period t

C_r^t = per unit reduction cost in period t.

The definition of other variables and costs remain the same.

We assume that $C_e^t + C_r^t \geq 0$ for otherwise the problem is unbounded.

Let $S_t = f X_t$, $U_t = f W_t$ and $V_t = f Z_t$ for $t = 1,2,...,T$.

Now the problem becomes

$$\text{(P7)} : \quad \text{Min} \quad (1/f) \sum_{t=1}^{T} [\, C_0 S_t + C_e^t U_t + C_r^t V_t + f (C_v - C_p) Y_t \,]$$

subject to

$$Y_t - S_t + g_t = 0 , \qquad t = 1,2,...,T \qquad (9)$$

$$S_t - S_{t-1} - U_t + V_t = 0 , \qquad t = 1,2,...,T \qquad (10)$$

$$Y_t \leq D_t , \qquad t = 1,2,...,T \qquad (11)$$

$$S_t, Y_t, U_t, V_t, g_t \geq 0 , \qquad t = 1,2,...,T \qquad (12)$$

where g_t, $t = 1,2,...,T$ are slack variables and $S_0 = f X_0$.

For $t = 1,2,\ldots,T-1$, subtracting the t th equation of (9) from the (t+1) th equation of (10), the constraints (9) and (10) are equivalent to:

$$Y_t - S_t + g_t = 0 , \quad t = 1,2,\ldots,T \tag{13}$$

$$S_1 - U_1 + V_1 = S_0 , \tag{14}$$

$$- Y_{t-1} + S_t - U_t + V_t - g_{t-1} = 0, \quad t = 2,\ldots,T \tag{15}$$

Each variable has atmost two non-zero coefficients in equations (13) to (15). Furthermore, if a variable has two non-zero coefficients, one of them is a +1 and the other is a -1. Consequently (P7) is a network flow problem with upper bounds on the variables Y_t, $t = 1,2,\ldots,T$ given by (11). Thus (P7) can be solved efficiently by using network flow algorithms.

Remark 8 : It can easily be shown that the dual of (P7) is also a network flow problem with upper bounds.

Remark 9 : For every t, atmost one of U_t and V_t can be positive in an optimal solution.

Next we study the structure of an optimal solution.

Adding the equations (13) to (15), we have

$$- \sum_{t=1}^{T} U_t + \sum_{t=1}^{T} V_t + Y_T + g_T = S_0$$

Now, letting $R = \sum_{t=1}^{T} V_t + Y_T + g_T$, the above redundant equation may be written as follows:

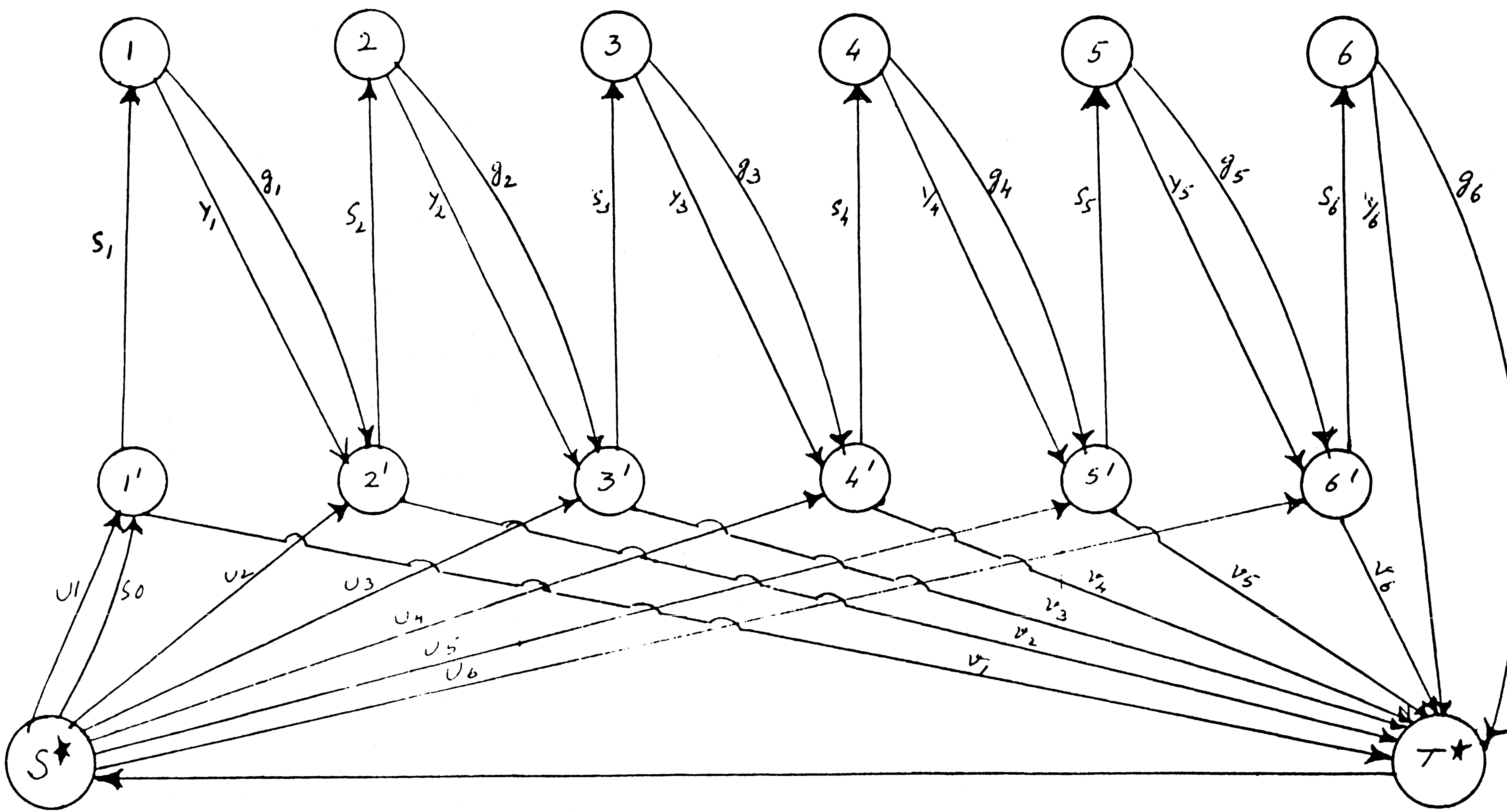

Figure 1 : Flow Circulation Network : Dynamic Version

$$\sum_{t=1}^{T} U_t - R = - S_0 \qquad \text{and} \tag{16}$$

$$R - \sum_{t=1}^{T} V_t - Y_T - g_T = 0 \tag{17}$$

Now the flow circulation form of (P7) is given by (11) to (17). The network corresponding to this flow circulation form is illustrated in Figure 1 for T=6.

Lemma 3 : In every extreme point of the polyhedron defined by the constraints (11) to (17),

i) if $U_t > 0$, then $S_t = D_j$ for some j such that $t \leq j \leq T$ and

ii) if $V_t > 0$, then $S_t = 0$ or $S_t = D_j$ for some j such that $t \leq j \leq T$.

Proof : i) Let U_t^*, V_t^*, Y_t^*, S_t^*, g_t^*, t = 1,2,...,T be an extreme point. Suppose the Lemma is false. Then there exists a k such that $U_k^* > 0$ and $S_k^* \neq D_j$ for any j, where $k \leq j \leq T$. There are two cases to consider.

Case 1 : $U_t^* = V_t^* = 0$, t = k+1,k+2,...,T.

Now $S_t^* = S_k^*$ for t = k+1,k+2,...,T. Consider any period t, where $k+1 \leq t \leq T$. If $Y_t^* = S_t^*$, then $Y_t^* < D_t$ and Y_t is not at its upper bound. If $Y_t^* = D_t$, it follows that $g_t^* > 0$. Thus the variables R, U_k, S_t, t = k,k+1,...,T and either Y_t or g_t, t = k,k+1,...,T (depending upon whether $Y_t^* < D_t$ or $Y_t^* = D_t$), are strictly positive and less than their corresponding upper bounds, if any. The arcs representing these variables form a cycle in the network corresponding to the flow circulation form of (P7). Consequently these variables cannot be strictly positive in any extreme point of the polyhedron defined by the constraints (11) to (17).

Case 2 : U_t^* or $V_t^* > 0$ for some t such that $k+1 \le t \le T$.
If $t \neq k+1$, let p be the smallest index such that U_p^* or $V_p^* > 0$ and $U_t^* = V_t^* = 0$ for $k < t \le p-1$. If $t = k+1$, let $p = k+1$. Suppose $V_p^* > 0$. Now the variables R, U_k, V_p, S_t, $t = k,k+1,\ldots,p-1$ and either Y_t or g_t, $t = k,k+1,\ldots,p-1$ (depending upon whether $Y_t < D_t$ or $Y_t = D_t$), are strictly positive and less than their corresponding upper bounds, if any. If $U_p^* > 0$, the variables U_k, U_p, S_t, $t = k,k+1,\ldots,p-1$ and either Y_t or g_t, $t = k,k+1,\ldots,p-1$ (depending upon whether $Y_t < D_t$ or $Y_t = D_t$) are strictly positive and less than their corresponding upper bounds, if any. In either case the corresponding arcs form a cycle in the network representing the flow circulation form of (P7). Consequently these variables cannot be strictly positive in any extreme point of the polyhedron defined by the constraints (11) to (17).

ii) The proof for this part is similar to the proof of part (i) and the details are omitted.

Remark 10 : If the objective function is a concave function, then there exists an extreme point optimal solution , see Hadley [2]. However, a local optimal solution is not a global optimal solution and an enumeration of the extreme points is required. Now Lemma 3 can be used to derive an efficient dynamic programming algorithm. The stages would correspond to the time periods and the states would represent the warehouse capacity at the beginning of each period. By Lemma 3, the number of states at each stage would be T+2 corresponding to a warehouse capacity of 0, S_0 and D_t, $t = 1,2,...,T$. The concave cost version of the problem is a generalization of the problem considered by Manne and Veinott [4]. In [4], the demands are non-decreasing over time and reduction of warehouse capacity is not permitted.

6. CONCLUSION

In this paper, a simple method of obtaining the optimal private warehouse size for the static problem is given. Four extensions of the static problem and their solutions are presented. The dynamic warehouse sizing problem is shown to be a network flow problem which could be solved easily using network flow algorithms. The concave cost version of the dynamic warehouse sizing problem can be solved efficiently using dynamic programming.

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