Working Paper 218

Endogenous Growth Cycles in Continuous Time

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December 2003

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ABSTRACT

The endogenous dynamics of a closed constant returns multi-market economy are examined in which agents face downward sloping demand. The trigger for growth in this model is a technological change that warrants costly adjustment in input quantities by agents. In the resulting dynamic game, relative prices within markets remain constant. Consequently, all own price elasticities are constant. In markets characterized by lower cost of capital the unique outcome is collusion in which agents do not incur adjustment cost and there is no adoption of new technology. But in other markets a unique non-cooperative equilibrium exists in which agents do incur the cost of adopting the new technology. Only three specifications of adjustment cost are feasible. Output increases along an S-shaped time path with or without a non-explosive cyclical component.

Keywords: Technological change; endogenous growth; business cycles

JEL classification: E32; O33; O41

Short Title: Endogenous Growth Cycles

1 Introduction

In an earlier article in this journal (Sinha, 1997), I had shown how investment in productivity by firms in an economy in which products were weak gross substitutes could lead to S-shaped time paths of output in the case of quadratic adjustment cost and had pointed out that other specifications of adjustment cost could result in cyclical time paths (p. 8). This paper extends that model in three ways. One, I relax the assumption of weak gross substitutability. Two, I explicitly model cost of capital to highlight the role it plays in the adoption of new technology. And three, I show that only three specifications of adjustment cost are feasible and that the shortest time path between two levels of output has an S-shape with a cyclical component.

My model is one of endogenous growth cycles. The time paths of output may be cyclical, but there are no recessions in this model of a closed economy with the dollar gross national product (GNP) held constant. Endogenous growth is the result of conscious economic choices made by agents. It occurs due to technological progress involving a shift of the isoquant map such that less input is needed for any output. Samuelson (1965) had shown that the competitive solution of directing technological progress to reducing unit cost was not optimal. Early models, therefore, analysed either a planned, or a competitive economy at the aggregate level, relying on externalities due to investment in education, research, knowledge, and the like to drive growth (Nordhaus, 1967; Uzawa, 1965; Phelps, 1966). Following Romer (1986), recent models exploit Dixit and Stiglitz's (1977) framework of monopolistic competition in which own price elasticities of firms are identical and constant. These models rely on either increasing returns disguised in the form of learning-by-doing à la Arrow (1962), specialization, or such other, or an externality due to either education or research to generate growth. Aghion and Howitt (1998) provide a recent review of this work.

My model differs from the above in five ways. One, I model a multimarket economy with asymmetric, monopolistically competitive agents (monopolists for short). Own price elasticities do turn out to be constant but because of a constraint in equilibrium. Two, I explicitly assume constant returns. Three, I model the cost of capital and show that growth results from non-cooperation among monopolists with high cost of capital. Four, I allow agents to enhance productivity in a manner they choose—by investing in quality control, value engineering, business process re-engineering, and the like. And five, I permit households to invest part of their savings into the development of new markets. Agents in these markets can exist only by helping buyers reduce cost of their current output or utility. This makes endogenous technological progress feasible and helps the economy sustain growth.

The trigger for growth is a productivity enhancing change in technology that, if adopted, will result in a different optimal input mix for agents. Adoption is not costless and gives rise to the problem of inter-temporal substitution. In the resulting dynamic equilibrium, agents in markets with lower cost of capital collude and do not incur adjustment cost, but agents in other markets do. The resulting growth is endogenous and is the outcome of competition among monopolistic agents.

Before I describe the general model let me provide a sketch. Consider a closed constant returns economy with dollar GNP m comprising n identical single product firms facing constant elasticity demand, each with price p and marginal cost $[c =]\ z\varsigma$, z and ς are quantity and price indexes of inputs. At $t = T$ a more productive technology becomes feasible. It will reduce z to $\underline{z}$ and p to $\underline{p}$, but is costly to adopt. In the non-cooperative equilibrium firms do reduce z to $\underline{z}$ by $t = \tau$. Let $\varphi \equiv -\dot{z}/z[\geq 0]$. It turns out that $(\varphi^*)^\lambda$ is constant. Thus, z has multiple roots. Three values of λ are feasible. For $\lambda = 2$, z has two real roots and for $\lambda = 4$ and $\lambda = 4/3$, two real and two complex roots with no real parts. The latter case yields the following time path for p:

$$p = \begin{cases} p_{1T}\exp(-\Lambda(t-T)) + p_{2T}\exp(\Lambda(t-T)) + p_{3T}\sin(\delta_T + \Lambda(t-T)) \text{ for } t \leq \tau, \\ \underline{p} \text{ otherwise.} \end{cases} \quad (1)$$

In (1), Λ is constant. The resulting time path of output, $q[= m/p]$, exhibits the well known S-shape with a cyclical component superimposed over it.

Economics has a rich tradition of use of discrete-time models that yield complex dynamics more easily. But, following Flaschel, Franke and Semmler (1994, p. 10), I work in continuous time. It facilitates mathematical analysis, yields new insight into what Frisch (1933) called the "impulse" and "propagation mechanism" of cycles, and allows me to go beyond his essentially static concept of equilibrium (steady state or damped oscillation of a pendulum). The S-shaped time path of output is due to the first two roots and cycles are due to the oscillating roots in (1), each linked to distinct input bundles. In the next two sections I analyse a particular episode of technological change. In section 4 I discuss the case when a series of such episodes leads to sustained growth.

2 DYNAMIC RESTRICTIONS ON PRICES

I consider monopolistically competitive agents—firms and households—in a closed economy comprising N markets indexed by $j(=1,\cdots,N)$ that include service and manufacturing industries in the service, consumption and capital goods sectors, occupations, and the security markets. I treat firms and households nearly alike and will point out the differences as and when they arise.

2.1 Market Clearing

At every t there exists a known unique market clearing price vector $\tilde{p} \equiv [\{p_1^1,\cdots,p_j^u,\cdots, p_N^{n_N}\} \in \mathbb{R}_+^n; n = \sum n_j]$. In addition to the standard assumptions in monopoly theory relating to the existence of inverse demand (Mas-Collel, Whinston and Green, 1995, p. 385), I assume that:

(a1) the capital market is perfect, financial markets are complete and there is no arbitrage,

(a2) preferences are homothetic and separable so that preference rankings of goods and services within market j are independent of the level of goods and services consumed in other markets,

(a3) technology exhibits constant returns, and

(a4) all agents in a market have access to identical technology.

The first assumption requires (1) divisibility of assets, (2) absence of transaction costs, taxes and restrictions on borrowings at the constant riskfree rate $r_f(>0)$, and (3) a belief by investors that they cannot influence the probability distribution of returns on securities.

From (a2), income elasticities are unitary and I can do all analysis with the dollar GNP, m, held constant. There are n_j firms indexed by $u(=1,\cdots,n_j)$ in industry j. Firm u sells one product or service at p_j^u, employs labour and other inputs, pays all profit as dividend which, for modelling convenience, I assume is in the form of share buybacks, and maximizes present value using a convex production or service technology. Its marginal cost at t is c_j^u and quantity demanded $q_j^u(\tilde{p},t)$. In case u uses part of output as input, $q_j^u(\tilde{p},t)$ is the net quantity demanded.

Household u in occupation j supplies one type of labour at p_j^u, consumes goods and services, receives dividends and maximizes wealth that equals discounted present value of savings and investments using convex consumption and investment technologies. Its marginal consumption expenditure at t is c_j^u, quantity of labour demanded $q_j^u(\tilde{p},t)$ and wealth $\Omega_j^u(\tilde{p},t)$.

There are S security markets indexed by $i(=1,\cdots,S)$. In security market i there are n_i securities indexed by $v(=1,\cdots,n_i)$. As I will show shortly, for security v in security market i—which is the common stock of firm u in industry j—$q_i^v(\tilde{p},t)$ is the price at t, p_i^v the rate of return—a random variable, and c_i^v the marginal cost that equals r_f. I denote the expected value of p_i^v by $r_j^u[\equiv E\{p_i^v\}]$; r_j^u is the instantaneous percentage change in the stock price of u and is assumed to be constant. In a stochastic model, with security prices following a process described by the geometric Brownian motion as in Merton (1971), r_j^u would be the constant drift term. Investors buy securities of old and new firms. Investment in new firms results in entry in new and established industries. I assume that investment in new industries is positive, so that $\dot{N}>0$.

Together with (a3), (a2) allows two-stage budgeting. Firms and households allocate budget to markets, then buy on price. The GNP share of market j is b_j which may vary over time due to changes in tastes and technology, formation of new markets, and the like; $\dot{b}_j/b_j\equiv-\beta_j$. The b_j s add up to unity if summed over either the goods and services markets or factor markets. Entry, changing tastes and technology, and formation of new industries influence the quantity demanded of firm u over time. Entry in occupation, emergence of new occupations, changing technology, and the like may influence the quantity of labour demanded of household u over time. Thus, $\partial q_j^u(\tilde{p},t)/\partial t=-\beta_j^u$. Own and cross price elasticities of demand of u are $\varepsilon_j^u(\tilde{p})[\equiv -(p_j^u/q_j^u(\tilde{p},t))(\partial q_j^u(\tilde{p},t)/\partial p_j^u)>1]$ and $\varepsilon_{ji}^{uv}(\tilde{p})[\equiv-(p_i^v/q_j^u(\tilde{p},t))(\partial q_j^u(\tilde{p},t)/\partial p_i^v)]$. Although cross price elasticities across markets are zero, I will write q_j^u and ε_j^u as $q_j^u(\tilde{p},t)$ and $\varepsilon_j^u(\tilde{p})$.

For single output agents, (a3) implies linearly homogeneous technology allowing me to write $c_j^u[=z_j^u\varsigma_j]$ as the product of quantity index, z_j^u, and price index, ς_j, of inputs; $z_j^u[\equiv z_j^u(x_j^1,\cdots,x_j^i,\cdots x_j^N)]$ is a function of quantities and $\varsigma_j[\equiv\varsigma_j(p_1,\cdots,p_i,\cdots,p_N)]$ of prices; x_j^i is the average quantity of input per unit of output used by agents in market j from market i and p_i the price in market i; $\dot{\varsigma}_j/\varsigma_j\equiv-\phi_j$. From (a4), ς_j is same across agents in market j. It is also beyond their control. In the security market i, $c_i^v=r_f$, which is same for all securities. But, I allow productivity differentials in other markets, that is, $z_j^u\neq z_j^v$; z_j^u is the reciprocal of productivity. A mechanism to account for this is as follows. When u enters market j, it incurs an entry cost equal to the expected present value of profits (savings for households) for its chosen levels of non-price parameters such as scale of entry, image, and the like, and draws a productivity parameter ζ_j^u from a lottery with known distribution. If luck favours u in the draw, ζ_j^u will be greater than the mean for the market. If not, u's unit cost will be above average. The market value of u will adjust immediately after the draw to reflect its actual relative cost position in market j.

At $t<T$ the closed, convex, and non-empty production set of firm u is given by $\underline{Y}_j^u=\{(-x_j^1,\cdots,-x_j^i,\cdots,-x_j^N,q_j^u):q_j^u-\underline{f}_j^u(x_j^1,\cdots,x_j^i,\cdots,x_j^N)\le 0 \text{ and } (x_j^1,\cdots,x_j^i,\cdots,x_j^N)\ge 0\}$, where $\underline{f}_j^u[\equiv\zeta_j^u\underline{f}_j(x_j^1,\cdots,x_j^N)]$ is u's production function and $\underline{f}_j$ that of an average competitor. $\underline{Y}_j^u$ satisfies the property of free disposal. At $t<T$, u faces no intertemporal tradeoff and solves:

$$\text{Maximise}_{f_j^u\ge q_j^u\ge 0}\{p_j^u(q_j^u)q_j^u(\tilde{p},t)-C_j^u(q_j^u)\}. \tag{2}$$

C_j^u is total variable cost; $\partial C_j^u/\partial q_j^u\,[\equiv c_j^u]$ is marginal cost. For household u, $\underline{Y}_j^u$ is consumption set; $\underline{f}_j^u$ and $\underline{f}_j$ are consumption functions for u and its average competitor. Household u solves:

$$\text{Maximise}_{f_j^u\ge q_j^u\ge 0;\omega_1^1,\cdots,\omega_S^{n_S}} E\{U_j^u(p_j^u(q_j^u)q_j^u(\tilde{p},t)-C_j^u(q_j^u)+\textstyle\sum_{i=1,\cdots,S}\sum_{v=1,\cdots,n_i} p_i^v\omega_{ij}^{vu}\Omega_j^u)\}. \tag{3}$$

The nature of $U_j^u(\cdot)$ will depend on the risk preference of $u;\omega_{ij}^{vu}$ is the fraction of u's wealth invested in security v in security market i; $\sum_{i=1,\cdots,S}\sum_{v=1,\cdots,n_i}\omega_{ij}^{vu}=1$. In this paper I do not analyse the case where p_j^u would be a random variable in a market other than the security markets.

Maximisation of (2)-(3) with respect to q_j^u yields the familiar equality of price-cost margin and the reciprocal of own price elasticity of demand, so that,

$$(p_j^u-c_j^u)/p_j^u=1/\varepsilon_j^u(\tilde{p}); \tag{4}$$

$$\{p_j^u q_j^u*(\tilde{p},t)-C_j^u(q_j^u*)\}=p_j^u q_j^u*(\tilde{p},t)/\varepsilon_j^u(\tilde{p}). \tag{5}$$

At t, to buy one unit of security v in security market i—which is the share of firm u in industry j, investors borrow $q_i^v(\tilde{p},t)$ at $c_i^v[=r_f]$. They expect it to be worth $(1+p_i^v)q_i^v(\tilde{p},t)$ at the end of the period from which they must return $(1+c_i^v)q_i^v(\tilde{p},t)$ to creditors. Their problem is:

$$\text{Maximise}_{q_i^v\ge 0}\ E\{p_i^v(q_i^v)q_i^v(\tilde{p},t)-c_i^v q_i^v(\tilde{p},t)\}. \tag{6}$$

A comparison of (6) and (2)-(3) makes it clear that the price of security v in security market i traded at t plays the role of $q_i^v(\tilde{p},t)$, and r_j^u and r_f play the roles of respectively $E\{p_i^v\}$ and c_i^v. Here r_j^u is the total expected return comprising dividends in the form of share buybacks as well as increases or decreases in u's value due to price and non-price factors. Maximisation yields

$$\varepsilon_i^v(\tilde{p}) = r_j^u/(r_j^u - r_f) \equiv e_j^u(\tilde{p}). \qquad (7)$$

With r_j^u and r_f constant, investors' own price elasticity, $e_j^u(\tilde{p})$, is constant. Since $\partial e_j^u(\tilde{p})/\partial r_j^u[= -r_f/(r_j^u - r_f)^2] < 0$, a smaller $e_j^u(\tilde{p})$ in (7) indicates a larger r_j^u. From homogeneity of demand,

$$\begin{bmatrix}
(\varepsilon_1^1-1) & \cdots & \varepsilon_{11}^{1n_1} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\varepsilon_{11}^{n_1 1} & \cdots & (\varepsilon_1^{n_1}-1) & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & (\varepsilon_j^1-1) & \cdots & \varepsilon_{jj}^{1n_j} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \varepsilon_{jj}^{n_j 1} & \cdots & (\varepsilon_j^{n_j}-1) & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & (\varepsilon_N^1-1) & \cdots & \varepsilon_{NN}^{1n_N} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & \varepsilon_{NN}^{n_N 1} & \cdots & (\varepsilon_N^{n_N}-1)
\end{bmatrix}
\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (8)$$

The existence of a (locally) unique market clearing price vector implies that the rank of the Jacobian of demand in market *j*, that is, the rank of the $n_j \times n_j$ block corresponding to market *j* in the $n \times n$ matrix of terms on the left hand side of (8), is $(n_j - 1)$ (Dierker, 1972).

2.2 *Investment in Productivity*

At $t = T$ firms in industry *j* become aware of a new productivity enhancing technology. If fully adopted, it will reduce *u*'s input quantity index $z_j^u[= \bar{z}_j^u$ at $t \le T]$ to $\underline{z}_j^u[< \bar{z}_j^u]$ and bring down unit cost in which case the new production set for $u = 1, \cdots, n_j$ would be:

$\bar{Y}_j^u = \{(-x_j^1, \cdots, -x_j^i, \cdots, -x_j^N, q_j^u) : q_j^u - \bar{f}_j^u(x_j^1, \cdots, x_j^i, \cdots, x_j^N) \le 0$ and $(x_j^1, \cdots, x_j^i, \cdots, x_j^N) \ge 0\}$;
where $\bar{f}_j^u(x_j^1, \cdots, x_j^i, \cdots, x_j^N) = \zeta_j^u \bar{f}_j(x_j^1, \cdots, x_j^i, \cdots, x_j^N)$ and, for all $(x_j^1, \cdots, x_j^i, \cdots, x_j^N) > 0$, $\bar{f}_j(x_j^1, \cdots, x_j^i, \cdots, x_j^N) > \underline{f}_j(x_j^1, \cdots, x_j^i, \cdots, x_j^N)$.

Adoption is costly. The advent of new technology, therefore, may result in a one-time drop in the value of firms in industry *j* at $t = T$. Define $\xi_j^u(\tilde{p}) \equiv -\dot{z}_j^u$ and $\varphi_j^u = \xi_j^u / z_j^u$. During *dt* at *t*, *u* can change z_j^u by $-\dot{z}_j^u$ if it spends a fraction $\varepsilon_j^u(\tilde{p})(\varepsilon_j^u(\tilde{p}) - 1)\Theta_j^u(\xi_j^u / z_j^u)$ of gross profit on productivity; $\Theta_j^u = 0$ if $\xi_j^u = 0$ and $\Theta_j^u > 0$ otherwise, so that $\xi_j^u \ge 0$. Else rivals would gain market share without incurring any cost. The term $\varepsilon_j^u(\tilde{p})(\varepsilon_j^u(\tilde{p}) - 1)$ will simplify algebra. In general, $\varepsilon_j^u(\tilde{p})$ need not be constant. It may change due to changes in buyers' tastes or if *u* repositions itself in the market. The latter exercise nearly amounts to new entry and may be about as expensive. In this paper I do not consider these cases. But I do allow the possibility of the new technology influencing $\varepsilon_j^u(\tilde{p})$ through its impact on relative costs and prices in market j.

The state variable is p_j^u; $\xi_j^u(\tilde{p})$ is control. For u, $\xi_j^u(\tilde{p}) \in \Xi_j^u$ is a pure closed loop strategy; Ξ_j^u, its strategy space, is Lipschitz continuous, that is, $\Xi_j^u = \{\xi_j^u(\tilde{p}) \mid \forall \xi_j^u(\tilde{p}) \in \mathbb{R}$; $\xi_j^u(\tilde{p})$ is continuous in $\tilde{p}$ and $|\xi_j^u(\tilde{p}) - \xi_j^u(\tilde{\psi})| \leq A \| \tilde{p} - \tilde{\psi} \|$ for some $A > 0$, for all $\tilde{p}, \tilde{\psi} \in \mathbb{R} \times \cdots \times \mathbb{R}\}$. $\Theta_j^u \geq 0$, $\partial \Theta_j^u / \partial \xi_j^u > 0$ and the Legendre condition requires that $\partial^2 \Theta_j^u / \partial \xi_j^{u2} > 0$.

Like u, v can also change z_j^v. Since $\varepsilon_{ji}^{uv}(\tilde{p}) = 0$ if $i \neq j$, I limit my analysis to industry j. Firm u takes as given $\xi_j^{v*}, v = 1, \cdots, n_j,\ v \neq u$ and δ_j^u, the rate at which investors discount a constant cash flow with risk characteristics similar to those of u—δ_j^u is a component of u's total cost of capital, r_j^u. It chooses $q_j^u(\tilde{p}, t)$ and $\xi_j^u(\tilde{p})$ to maximize present value of profits. Ignoring fixed cost and depreciation, $V_j^u(\tilde{p}, t)$, the twice continuously differentiable optimal value of u's objective functional at $t[\geq T]$ is given by:

$$V_j^u(\xi_j^u) = \underset{\bar{f}_j^u \geq q_j^u \geq 0; \xi_j^u \geq 0}{\text{Max}} \int_t^\infty \exp\{-\delta_j^u(s-t)\}\{p_j^u(q_j^u)q_j^u(\tilde{p},t)$$

$$-C_j^u(q_j^u)\}\{1 - \varepsilon_j^u(\tilde{p})(\varepsilon_j^u(\tilde{p}) - 1)\Theta_j^u(\xi_j^u / z_j^u)\}ds \qquad (9)$$

subject to $\dot{p}_j^u = \Psi_j^u(\tilde{p}, \xi_j^u)$ and $p_j^u(T) = p_{jT}^u$.

The argument of $V_j^u(\tilde{p}, t)$ in (9) indicates that it is to be maximized with respect to ξ_j^u, although it depends on δ_j^u and the strategy n_j-tuple $(\xi_j^1(\tilde{p}), \cdots, \xi_j^{n_j}(\tilde{p}))$ over $\mathbb{R} \times \cdots \times \mathbb{R}$ of competitors.

2.3 The Nature of Adjustment Costs

A trivial solution of (9) with $\xi_j^u = 0$ for all u, j, is possible. It represents the unique equilibrium in cases of perfect competition and collusion. Perfect competitors have no resources to invest in productivity and firms maximizing joint profit no incentive. Nontrivial solutions may exist in the case of non-cooperation. In repeated price games non-cooperation is known to be optimal for large discount rates (Friedman, 1971). Thus, r_j^u may determine φ_j^u in equilibrium. If u invests too little in productivity, investors will increase the discount rate. The fall in value will signal u that it must increase investment in productivity. Since I seek a non-trivial solution, I assume that $\xi_j^u \neq 0$. During dt at t, u invests $I_j^u dt[\equiv (p_j^u q_j^{u*}(\tilde{p},t) / \varepsilon_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\Theta_j^u(\varphi_j^u(r_j^u))dt \geq 0]$ in productivity.

Firm u can either borrow, or use own funds, or raise new equity. The mode of financing will not affect its value (Modigliani and Miller, 1958). Suppose it uses own funds. Dividends will go down by $I_j^u dt$. But shareholders do not care since they can sell that amount of u's shares.

Now consider the owners of u's shares trading as security v in security market i. They take as given $(\xi_j^1*,\cdots,\xi_j^{n_j}*)$ and sell u's shares valued at $I_j^u dt$ for a rate of return of $p_i^v; E\{p_i^v\}=r_j^u$.

As in (4), their problem is:

$\text{Maximise}_{I_j^u \geq 0} E\{I_j^u(\varphi_j^u*(r_j^u))(p_i^v - r_f)dt\}$.

At the optimal,

$$e_j^u[\equiv -(r_j^u / I_j^u*)(\partial I_j^u* / \partial r_j^u)] = -(r_j^u \Theta_j^{u\prime} / \Theta_j^u)(\partial \varphi_j^u / \partial r_j^u) = r_j^u / (r_j^u - r_f). \quad (10)$$

Equation (10) represents the productivity implication of equilibrium in the capital markets. For given $(r_j^1*,\cdots,r_j^{n_j}*)$, a Nash equilibrium of (9) is a strategy n_j-tuple $(\xi_j^1*,\cdots,\xi_j^{n_j}*)$ such that, for $u=1,\cdots,n_j$,

i. $\xi_j^u* \in \Xi_j^u$, and

ii. $V_j^u(\xi_j^1*,\cdots,\xi_j^{u-1}*,\xi_j^u*,\xi_j^{u+1}*,\cdots,\xi_j^{n_j}*) \geq V_j^u(\xi_j^1*,\cdots,\xi_j^{u-1}*,\xi_j^u,\xi_j^{u+1}*,\cdots,\xi_j^{n_j}*)$.

Proposition 1: In the unique Nash equilibrium of the game in (9), for $t \geq T$ and for all u, v, j, $\Theta_j^u(\xi_j^u / z_j^u)$ takes the form of $[\theta_j^u(\xi_j^u / z_j^u)^{\lambda_j^u} \equiv]\theta_j^u(\varphi_j^u)^{\lambda_j^u}$ where $\theta_j^u[>0]$ and $\lambda_j^u[>1]$ are constant;

$$(\varphi_j^u)^{\lambda_j^u} = \text{Max}\{0,(\lambda_j^u - e_j^u(\lambda_j^u - 1))/(\lambda_j^u + e_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u\}; \quad (11)$$

$\dot{p}_j^u / p_j^u = \dot{p}_j^v / p_j^v$; $\dot{\varepsilon}_j^u = 0$; and the time path of $z_j^u \in \bar{Y}_j^u$, $u=1,\cdots,n_j$, is given by

$$\dot{z}_j^u / z_j^u[= \dot{z}_j^v / z_j^v] = -\varphi_j* = -\varphi_j^u* = -\varphi_j^v* = \text{Min}\{-\varphi_j^1,\cdots,-\varphi_j^{n_j},0\}. \quad (12)$$

Proof: See Appendix.

Thus, even monopolists cannot change prices at will. If competitors of u reduce price and it fails to reduce p_j^u at the same rate, it loses market share and its value goes down. The rate of change of prices plays the same role for monopolists that price does for perfect competitors.

3 NON-EXPLOSIVE DYNAMICS

Proposition 1 does not rule out the trivial solution. I now derive the conditions under which a unique non-cooperation equilibrium exists and agents invest in productivity.

3.1 Cost of Capital and the Adoption of New Technology

The production set requiring $(-x_j^1, \cdots, -x_j^N, q_j^u) \in \bar{Y}_j^u$ imposes restrictions on the values of λ_j^u.

Proposition 2: In (11), λ_j^u equals either 4/3, or 2, or 4.

Proof: See Appendix.

Several studies in the past have relied on a quadratic specification of adjustment costs (Hamermesh and Pfann, 1996). The specification resulting from Proposition 2 is more general and has two implications. One, the larger *u*'s gross profit, the more it must spend for any percentage reduction in z_j^u. And two, to double the rate of growth of productivity *u* must more than double the investment in productivity.

From (12) $\lambda_j^u = \lambda_j^v = \lambda_j$ since $\varphi_j^u = \varphi_j^v$. Since, $\partial\varphi_j^u / \partial\lambda_j < 0$, smaller values of λ_j are associated with higher rates of growth in productivity. This is intuitive since a higher λ_j implies a higher adjustment cost for the same value of φ_j^u. Thus, among the feasible values of λ_j the smallest one will yield the shortest time path of z_j^u between z_{jT}^u and $\underline{z}_j^u$ and will prevail.

Corollary 1: Let all values of λ_j be available. Firms in industry *j* will not adopt the new technology if $r_j^u \leq (4/3)r_f$ for all *u*. If $r_j^u > (4/3)r_f$ for some *u*, $r_j^u > (4/3)r_f$ for all *u*, $\lambda_j = 4/3$, and adjustment cost's share of gross profit $[= \varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u(\varphi_j *)^{\lambda_j}]$ never exceeds 3/7.

The collusive outcome prevails if the cost of capital of every firm in the industry is less than $(4/3)r_f$. In such an industry entry by a firm with $r_j^u > (4/3)r_f$ may facilitate adoption of new technology. The following results apply if the value of λ_j in the industry is constrained.

Corollary 2: Let λ_j be either 2 or 4. Firms in industry *j* do not adopt new technology if $r_j^u \leq 2r_f$ for all *u*. If $r_j^u > 2r_f$ for some *u*, $r_j^u > 2r_f$ for all *u*, $\lambda_j = 2$, and $\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u(\varphi_j *)^{\lambda_j} \leq 1/3$.

Corollary 3: Let $\lambda_j = 4$. Firms in industry *j* do not adopt the new technology if $r_j^u \leq 4r_f$ for all *u*. If $r_j^u > 4r_f$ for some *u*, $r_j^u > 4r_f$ for all *u*, and $\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u(\varphi_j *)^{\lambda_j} \leq 1/5$.

3.2 Feasible Time Paths

Let Λ_j be a real positive root of $\{(\lambda_j - e_j^u(\lambda_j - 1))/(\lambda_j + e_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u\}^{1/\lambda_j}$. For $\lambda_j = 4/3$, one could look at Λ_j as a real positive root of $\{[(\lambda_j - e_j^u(\lambda_j - 1))/(\lambda_j + e_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u]^3\}^{1/4}$. For $\lambda_j = 2$:

$$z_j^u = \begin{cases} z_{j1T}^u \exp(-\Lambda_j(t-T)) + z_{j2T}^u \exp(\Lambda_j(t-T)) \text{ for } t \le \tau_{jT}, \\ \underline{z}_j^u \text{ otherwise.} \end{cases} \quad (13)$$

For $\lambda_j = 4/3$ and $\lambda_j = 4$, z_j^u has two real and two complex roots with no real part and

$$z_j^u = \begin{cases} z_{j1T}^u \exp(-\Lambda_j(t-T)) + z_{j2T}^u \exp(\Lambda_j(t-T)) + z_{j3T}^u{}' \sin(\delta_{jT} + \Lambda_j(t-T)) \text{ for } t \le \tau_{jT}, \\ \underline{z}_j^u \text{ otherwise.} \end{cases} \quad (14)$$

Boundary conditions and the requirement in (11) that the time path of z_j^u be the shortest between z_{jT}^u and $\underline{z}_j^u$ for given parameters provide the values of the constants z_{j1T}^u and z_{j2T}^u in (13) and z_{j1T}^u, z_{j2T}^u, $z_{j3T}^u{}'$ and δ_{jT} in (14). In both cases, $\tau_{jT} = T + (1/2\Lambda_j)\ln(z_{j1T}^u / z_{j2T}^u)$. For $\lambda_j = 2$,

$z_{j1T}^u = \{z_{jT}^u + [(z_{jT}^u)^2 - (\underline{z}_j^u)^2]^{1/2}\}/2$, and

$z_{j2T}^u = \{z_{jT}^u - [(z_{jT}^u)^2 - (\underline{z}_j^u)^2]^{1/2}\}/2$.

For $\lambda_j = 4/3$ and $\lambda_j = 4$, it can be shown that $\delta_{jT} = \pi/2$;

$z_{j1T}^u = \{[z_{jT}^u + \ddot{z}_{jT}^u/(\Lambda_j)^2]/2 + [(z_{jT}^u + \ddot{z}_{jT}^u/(\Lambda_j)^2)^2/4 - (\underline{z}_j^u + (z_{jT}^u - \ddot{z}_{jT}^u/(\Lambda_j)^2)/2)^2]^{1/2}\}/2$,

$z_{j2T}^u = \{[z_{jT}^u + \ddot{z}_{jT}^u/(\Lambda_j)^2]/2 - [(z_{jT}^u + \ddot{z}_{jT}^u/(\Lambda_j)^2)^2/4 - (\underline{z}_j^u + (z_{jT}^u - \ddot{z}_{jT}^u/(\Lambda_j)^2)/2)^2]^{1/2}\}/2$, and

$z_{j3T}^u{}' = \{z_{jT}^u - \ddot{z}_{jT}^u/(\Lambda_j)^2\}/2$.

Figure 1(a) shows the plots of each component of z_j^u along with the time path of their sum for $\lambda_j = 2$. The time paths extend beyond $t = \tau_{jT}$. The horizontal line at $\underline{z}_j^u$ in the figure represents the transformation frontier. The time path of the sum of the components declines, touches the transformation frontier tangentially at $t = \tau_{jT}$ and then turns upward.

Insert Figure 1(a) about here

I can write $z_{j3T}^u{}' \sin(\delta_{jT} + \Lambda_j(t-T))$ in (14) as $\{z_{j3T}^u \sin(\Lambda_j(t-T)) + z_{j4T}^u \cos(\Lambda_j(t-T))\}$ for $\lambda_j = 4/3$ and $\lambda_j = 4$. Figure 1(b) shows plots of the four components of z_j^u and their sum. Again the time path of the sum declines, touches the transformation frontier tangentially and turns upward. This will not happen for any value of λ_j other than the three in Proposition 2.

Insert Figure 1(b) about here

3.3 Stability

The process leading to equilibrium in Proposition 1 also guarantees its local asymptotic stability.

Proposition 3. The trajectory of the system $\dot{p}_j^u / p_j^u = -\Phi_j^u, u = 1,\cdots,n_j; j = 1,\cdots,N$, converges to $\tilde{p}^* \equiv \{p_1^1*,\ldots,p_j^u*,\ldots,p_N^{n_N}*\}$.

Proof: See Appendix.

4 DISCUSSION AND IMPLICATIONS

The previous two sections investigated one episode of technological change. In this section I consider an economy undergoing a series of such changes. I describe time paths of variables, discuss their correspondence with stylized facts, provide an interpretation of the mechanism of growth and list factors that help the economy sustain growth in the long run.

4.1 Time Paths of Prices, Wages and Output

Let a_{ji} be the share of inputs from market *i* in the price of agents in market *j*; $a_{jj} = 0$ and $\sum_i a_{ji} < 1$. Matrix $[a]$ below is nonsingular and Φ_j's, ϕ_j's and φ_j's are given by:

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jj} & \cdots & a_{jN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{Nj} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_N \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_j \\ \vdots \\ \phi_N \end{bmatrix} = \begin{bmatrix} \Phi_1 - \varphi_1 * \\ \vdots \\ \Phi_j - \varphi_j * \\ \vdots \\ \Phi_j - \varphi_N * \end{bmatrix}.$$

A price change in market *j* does not directly influence the demand facing agents in market *i* but may affect their cost and, therefore, price and quantity demanded since $[\Phi] = [I - a]^{-1}[\varphi^*]$.

Figure 2(a) shows time paths of P_j^u, P_j and *P* for $\lambda_j = 2$. $P[\equiv p]$ is price index and $P_j^u[\equiv p_j^u / p]$ and $P_j[\equiv p_j / p]$ are prices relative to *P*; $\dot{P}/P = -\Phi[= -\sum b_j \Phi_j$, the sum being taken over the goods and the services markets alone]. The figure also shows time paths of W_j^u, W_j, and *W* for a full employment economy undergoing sustained growth in per capita output with wage share of GNP constant which requires that $\dot{P}/P < 0$ and $\dot{W}/W > 0$. $W_j^u[\equiv w_j^u / P]$,

$W_j[\equiv w_j / P]$, and $W[\equiv w/P]$ are wages relative to P; w_j^u is the wage level of household u in occupation j, w_j is that in occupation j and w is the wage index. Let the Greek letter ν ("nu") denote growth rate of population; $\nu \geq 0$. Output, m/P, grows at $-\dot{P}/P$. In an economy with growing per capita output $-\dot{P}/P-\nu > 0$. If the wage share of GNP is constant, employment will grow at $-\dot{w}/w$. If it is also a full employment economy then $-\dot{w}/w=\nu$. Therefore, $\dot{W}/W[=\dot{w}/w-\dot{P}/P=-\nu-\dot{P}/P]>0$ as shown in Figure 2. Figure 2(b) shows these time paths for $\lambda_j = 4/3$ and $\lambda_j = 4$. The model, however, is capable of generating more complex dynamics than the nearly synchronous time paths shown in these figures.

Insert Figure 2(a) about here

Insert Figure 2(b) about here

Figure 3(a) shows time paths of q_j^u, q_j^v, q_j, q_i, and q for $\lambda_j = 2$. Outputs of firms and industries exhibit periods of growth, maturity and decline characteristic of the product life cycle (PLC-) stages; $\dot{q}_j^u/q_j^u = -\beta_j^u + \Phi_j$ and $\dot{q}_j/q_j = -\beta_j + \Phi_j$. Both β_j^u and β_j are positive in the figure which they will be in the long term since the output of a firm or an industry can never be bigger than that of the economy. The time path of q is non-decreasing since $\dot{q}/q[=\Phi] \geq 0$. For $\lambda_j = 4/3$ and $\lambda_j = 4$, the time paths in Figure 3(b) have an additional cyclical component.

Since prices and output incorporate influences of productivity growth in several markets, the time paths in Figure 2 and Figure 3 will exhibit cycles so long as $\lambda_j \neq 2$ in some markets.

Insert Figure 3(a) about here

Insert Figure 3(b) about here

4.2 Stylised Facts and the Evolution of Prices, Wages, Productivity and Output

Kydland and Prescott (1990) show that, in the U.S. during 1954-89, the price level has displayed a counter-cyclical pattern. Fiorito and Kollintzas (1994) report similar findings for the G7 countries during 1960-1989. Labour productivity is widely believed to be pro-cyclical. But Fiorito and Kollintzas find real wages procyclical in some countries and countercyclical in others.

Since there is no recession in our model, the pro- or counter- cyclicality of variables must be judged in relation to the rate of growth of output. Real prices in Figure 2(b) are countercyclical since $\dot{q}/q = -\dot{P}/P[=\Phi]$. Labour productivity is procyclical since Φ is a linear increasing combination of the rates of growth of productivity in different markets. In Figure 2(b) real wages are procyclical for the simple reason that the figure has been drawn for a closed full employment economy undergoing sustained growth in per capita output with wage share of GNP constant.

A number of firm and industry level studies have documented S-shaped time paths of output under the name of "diffusion-" or the "PLC-" curve (Gort and Klepper, 1982), explanations for which often rely on an exogenous limit to the size of the market coupled with either heterogeneity across adopters, or strategic interaction among them giving rise to heterogeneity of the adoption decision over time. Support for the S-shaped time path for economies comes from Cho (1994) who reports a "humped" pattern of growth of output in Japan, U.K., and U.S. over the past 100 years. He attributes it to the convergence hypothesis of Baumol, Blackman, and Wolff (1991).

Most economists studying aggregate level variables have generally focused their attention on the cyclicality and co-movement of price, wages, employment, productivity and output, explanations for which usually rely on either exogenous monetary shocks or random productivity shocks on competitive allocation often under rational expectations (Lucas, 1975; Long and Plosser, 1983), although there has been a renewed interest in the role of nominal rigidities due to menu costs, coordination failures, long term contracts, and the like (Mankiw, 1985; Blanchard and Kiyotaki, 1987).

4.3 The Mechanism of Growth

The mechanism of growth here is different. The Legendre condition for maximum results in multiple roots to u's solution to its optimisation problem. The requirement that the time paths lie within the production set for all t limits the number of roots to either two or four. Two roots result in S-shaped time paths, to which four roots add two complex roots with no real parts.

Consider an example for $\lambda_j = 2$. The production technology of u is $q = x_1^{\alpha} x_2^{1-\alpha}$. Prices of x_1 and x_2 are respectively $p_1[=\$200]$ and $p_2[=\$50]$. The marginal cost of u is $[c =]$ $(p_1/\alpha)^{\alpha}(p_2/(1-\alpha))^{1-\alpha}$. At $t < T$, $\alpha = 0.7$, so that $c \approx \$243.06$. Super- and sub- scripts here do not indicate agents or markets. Table 1 provides these numbers at a glance and Figure 4(a), which is not to scale, shows u's budgeting problem in the x-plane. Till $t < T$, its optimal allocation is given by point 1 on isoquant I, for which $q = 1; x_{1,0.7} \approx 0.851$, and $x_{2,0.7} \approx 1.458$.

Insert Table 1 about here

Insert Figure 4(a) about here

Insert Figure 4(b) about here

At $t = T$, a new technology becomes available for which $\alpha = 0.6$. If fully adopted, it will reduce u's marginal cost to about \$225.16. This technology is represented by isoquant II which is tangent to the budget line at point 2 for which $x_{1,0.6} \approx 0.68$ and $x_{2,0.6} \approx 1.801$. Thus u would like to reduce x_1 and increase x_2 until it reaches 2. If changing input levels is costless, u will adjust them instantaneously. Adjustment cost limits the rate of substitution. Technological progress here results from substitution opportunities and its rate is determined by adjustment costs relating to the substitution of one set of inputs by another.

The model transforms the problem from x-plane to z-plane. In the case of $(N-1)$ inputs, the transformation will be from $(N-1)$-dimensional x-space to two-dimensional z-plane in the case of two roots and to four-dimensional z-space in the case of four roots. In Figure 4(b) the transformation is such that the new technology is given by $q = Kz_1^{1/2}z_2^{1/2}$ where z_1 and z_2 are input bundles with identical prices. Points 1 and 2 in Figure 4(a) correspond to points 1 and 2 in Figure 4(b). In the z-plane u moves along 1-2, at every point of which z_1 and z_2 represent a partition of u's inputs such that u can reduce z_1 by some amount, increase z_2 by a smaller mount, and produce the same output. In equilibrium, z_2 goes up and z_1 down at the same rate until they are equal. As a result, $z[\equiv z_1 + z_2]$ monotonically goes down to reach $\underline{z}[= 2z_1^{1/2}z_2^{1/2}]$ at $t = \tau$, at which point substitution stops. Although the values of $x_{1,0.6}$ and $x_{2,0.6}$ are different, $z_{1,0.6} = z_{2,0.6} = \underline{z}/2$. To obtain numerical values I normalize prices of z_1 and z_2 to \$243.06 so that $\overline{z}[\equiv z_{1,0.7} + z_{2,0.7}] = 1$, and $\underline{z}[\equiv z_{1,0.6} + z_{2,0.6}] = 225.16/243.06 \approx 0.926$. Then, $z_{1,0.7} = 0.688[= (\overline{z} + (\overline{z}^2 - \underline{z}^2)^{1/2})/2]$, $z_{2,0.7} = 0.312[= (\overline{z} - (\overline{z}^2 - \underline{z}^2)^{1/2})/2]$, $z_{1,0.6} = z_{2,0.6} \approx 0.463$, and $K \approx 2.159$.

In a physical explanation of Figure 4(a) and Figure 4(b) x_1 and x_2 may be two types of labour—the former trained in manufacturing and the latter in quality control (QC). QC technology is underdeveloped at $t < T$, but an advanced version becomes available at $t = T$. Then, z_1 and z_2 are quantities of identically priced labour composites, the former comprising more manufacturing and the latter more QC labour; $q = Kz_1^{1/2}z_2^{1/2}$. If u increases z_2, there will be fewer goods returned and less reworking needed, so that, less of z_1 will be required. But that requires expenditure on hiring and firing, training and equipment. Additional units of z_2 progressively save less of z_1. The optimal strategy for u is to move from 1 in Figure 4(b) along the isoquant to 2 where $z_1 = z_2$. For $\lambda_j = 4/3$ and $\lambda_j = 4$, the oscillating roots may be input bundles agents use to periodically accelerate the declining rate of growth of productivity. For instance, project teams made of manufacturing and QC workers may identify and implement projects to accomplish targeted substitution of inputs. The rate of growth of productivity will rise as new projects begin and fall as they near completion. Then another project will be taken up, thus propagating the cycle. After reaching the final target members of the project team revert back to regular assignments.

4.4 Sustained Growth

Ever since Solow (1957), we have known that increases in labour and capital were insufficient to explain the observed sustained rates of growth of output in the long run in many economies. In his model, in the long run or the steady state, per capita output and capital stock grew at a uniform rate equal to an exogenous rate of growth of total factor productivity. In recent years economists have analysed mechanisms that generate such growth endogenously. This happens in Romer (1986), for instance, because of increasing returns to specialization.

Two factors guarantee sustained growth in our model: one, $\varphi_j{}^* \geq 0$ for all j in the presence of rivals; and two, since $\dot{N} > 0$, $\varphi_j{}^* > 0$ for some j for all t due to non-cooperation. Growth in Romer (1986) is also driven by an increase in the number of intermediate goods. But he relies on a production function with increasing returns to number of inputs, so that a new intermediate good always increases economy's output. In our model part of economy's savings is invested in projects that attempt to form new markets. Only a few will succeed. These will be ones that help non-cooperative buyers in other markets reduce cost. Thus are born new markets and new technologies that propel output upward. The adoption of a new technology will end in finite time. But newer technologies will then be developed and the process will be repeated.

The QC interpretation of the process of adoption described earlier is not the only one possible. Emergence of new media such as internet may offer firms opportunity for increasing advertising and promotion expenditure to build brand and image and lower unit distribution cost since retailers are willing to stock better known brands for a lower commission. Availability of more productive but expensive machinery similarly reduces unit cost. Substitution opportunities across inputs that reduce unit cost but are costly to implement are the essence of increased output in this model. A continuous flow of such opportunities allows an economy to sustain growth.

Policies to encourage investment in the formation of new markets may help growth, but incentives to agents for accelerating productivity growth may have an unintended effect. Potential output of u is determined by $\underline{z}_j^u$ and e_j^u influences speed of adoption of new technology; both are beyond control. Governments often use θ_j^u as lever by subsidising investment in productivity.

Large firms also encourage cost reduction by suppliers. Such policies may help marginal firms survive technological change, but may be dysfunctional if used for accelerating growth. There are diminishing returns to investment in productivity. For λ_j equal to 4, 2, and $4/3$, u must increase investment in productivity respectively [2^{λ_j} =] 16, 4, and (approximately) 2.52 times to double the rate of growth. But that may reduce funds for the formation of new markets.

5 CONCLUSION

This paper has reported five new results. One, a closed constant returns economy with monopolistic agents exhibits endogenous dynamics. Two, the role price plays in competitive equilibrium is played by the rate of movement of prices in monopolistic equilibrium. If rivals of an agent cut price, and it fails to reduce price at the same rate, it loses market share and value. Three, growth results from competition among monopolists with high cost of capital who alone have the resources and the incentive to incur adjustment cost. Neither perfect competition nor collusion results in growth. Four, adjustment costs take one of three forms. The resulting time path of output has an S-shape with or without a cyclical component. And five, if all adjustment technologies are feasible, the shortest time path between two levels of output is cyclical. It results in the highest rate of growth of productivity and the quickest adoption of new technology.

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Appendix

Proof of Proposition 1

The n_j Hamilton-Jacobi-Bellman equations satisfied by $V_j^u(\tilde{p},t)$, $u=1,\cdots,n_j$ are:

$$\delta_j^u V_j^u(\tilde{p},t) - V_t^u(\tilde{p},t) = \underset{\bar{f}_j^u \geq q_j^u \geq 0; \xi_j^u}{\text{Max}} \{(p_j^u q_j^u(\tilde{p},t) - C_j^u(q_j^u))(1-\varepsilon_j^u(\tilde{p})(\varepsilon_j^u(\tilde{p})-1)\Theta_j^u(\xi_j^u / z_j^u)) + V_u^u(\tilde{p},t)\Psi_j^u(\tilde{p},\xi_j^u) + \Sigma_{v\neq u} V_v^u(\tilde{p},t)\Psi_j^v(\tilde{p},\xi_j^v *) + \Gamma_j^u \xi_j^u\} \tag{A1}$$

subject to $\dot{p}_j^u = \Psi_j^u(\tilde{p},\xi_j^u)$ and $p_j^u(T) = p_{jT}^u$,

where $V_t^u \equiv \partial V_j^u / \partial t$, $V_u^u \equiv \partial V_j^u / \partial p_j^u$, $V_v^u \equiv \partial V_j^u / \partial p_j^v$; $v \neq u$, and Γ_j^u is the multiplier associated with the constraint $\xi_j^u \geq 0$. Competitors respond to changes in prices, not quantities. Maximisation of (A1) with respect to q_j^u, therefore, involves no intertemporal tradeoff, yielding (5)-(6) as before. Since $p_j^u = \varsigma_j z_j^u \varepsilon_j^u(\tilde{p})/(\varepsilon_j^u(\tilde{p})-1)$, the state equation becomes $\dot{p}_j^u = p_j^u(\eta_j^u(\tilde{p}) - \phi_j - \xi_j^u(\tilde{p})/z_j^u)$, where $\eta_j^u(\tilde{p}) \equiv d(\varepsilon_j^u(\tilde{p})/\varepsilon_j^u(\tilde{p})-1)/dt$. Substituting into (A1):

$$\delta_j^u V_j^u(\tilde{p},t) - V_t^u(\tilde{p},t) = \underset{\xi_j^u}{\text{Max}}\{(p_j^u q_j^u *(\tilde{p},t)/\varepsilon_j^u)(1-\varepsilon_j^u(\varepsilon_j^u-1)\Theta_j^u) + V_u^u(\tilde{p},t)(\eta_j^u - \phi_j - \xi_j^u / z_j^u)p_j^u + \Sigma_{v\neq u} V_v^u(\tilde{p},t)(\eta_j^v - \phi_j - \xi_j^v * / z_j^v)p_j^v + \Gamma_j^u \xi_j^u\} \tag{A2}$$

subject to $p_j^u(T) = p_{jT}^u$.

In (A2) I have ignored the arguments for $\eta_j^u(\tilde{p})$, $\varepsilon_j^u(\tilde{p})$ and $\Theta_j^u(\xi_j^u / z_j^u)$ to avoid clutter. Maximising it I get:

$$-p_j^u q_j^u *(\tilde{p},t)(\varepsilon_j^u - 1)(\partial\Theta_j^u / \partial\varphi_j^u)/z_j^u - V_u^u p_j^u / z_j^u = 0; \tag{A3}$$

$\Gamma_j^u \geq 0$; and

$\Gamma_j^u \xi_j^u = 0$.

Writing $\Theta_j^u{}'$ for $\partial\Theta_j^u / \partial\varphi_j^u$, the solution for (A3) for $u=1,\cdots,n_j$ is:

$$V_j^u(\tilde{p},t) = \Theta_j^u{}' p_j^u q_j^u *(\tilde{p},t)(1-\gamma_j^u) + \gamma_j^u p_j^u q_j^u *(\tilde{p},t)/\varepsilon_j^u r_j^u, \tag{A4}$$

where $\gamma_j^u = 1$ if $\xi_j^u = 0$ and $\gamma_j^u = 0$ otherwise; γ_j^u takes care of the value of $V_j^u(\tilde{p},t)$ at $\xi_j^u = 0$. I seek solutions with $\xi_j^u \neq 0$. Define $\Phi_j^u \equiv -\dot{p}_j^u / p_j^u$. From (A4), $V_t^u(\tilde{p},t) = -\beta_j^u V_j^u(\tilde{p},t)$ and

$$V_u^u(\tilde{p},t)\dot{p}_j^u + \Sigma_{v\neq u} V_v^u(\tilde{p},t)\dot{p}_j^v = \{(\varepsilon_j^u - 1)\Phi_j^u + \Sigma_{v\neq u}\varepsilon_{jj}^{vu}\Phi_j^v)\}V_j^u(\tilde{p},t) \tag{A5}$$

Let $\{(\varepsilon_j^u - 1)\Phi_j^u + \Sigma_{v\neq u}\varepsilon_{jj}^{vu}\Phi_j^v\} \equiv \rho_j^u$. Substituting into (A2), and after some algebra I get:

$$(\delta_j^u + \beta_j^u - \rho_j^u)p_j^u q_j^u *(\tilde{p},t)\Theta_j^u{}' = p_j^u q_j^u *(\tilde{p},t)\{1-\varepsilon_j^u(\varepsilon_j^u - 1)\Theta_j^u\}/\varepsilon_j^u. \tag{A6}$$

In (A6) $(\delta_j^u + \beta_j^u - \rho_j^u)$ equals $r_j^u *$, the total (expected) rate of return to investors due to cash flow and capital loss or gain due to changes in non-price influences on demand and price changes by u and competitors; u takes $r_j^u *$ as given. Thus, I can rewrite (A6) as

$$r_j^u * \Theta_j^{u\,\prime} = \{1 - \varepsilon_j^u(\varepsilon_j^u - 1)\Theta_j^u\} / \varepsilon_j^u . \tag{A7}$$

Differentiating (A7) with respect to $r_j^u *$ and writing $\Theta_j^{u\,\prime\prime}$ for $\partial\Theta_j^{u\,\prime}/\partial\varphi_j^u$, I get

$$\Theta_j^{u\,\prime} + r_j^u * \Theta_j^{u\,\prime\prime}(\partial\varphi_j^u / \partial r_j^u) = -(\varepsilon_j^u - 1)\Theta_j^{u\,\prime}(\partial\varphi_j^u / \partial r_j^u),$$

which, after some algebra, yields:

$$-(r_j^u * \Theta_j^{u\,\prime} / \Theta_j^u)\partial\varphi_j^u / \partial r_j^u = \Theta_j^{u\,\prime} / \Theta_j^u\{((\varepsilon_j^u - 1)/r_j^u *) + \Theta_j^{u\,\prime\prime} / \Theta_j^{u\,\prime}\} . \tag{A8}$$

In capital market equilibrium, the left side of (A8) equals e_j^u. From (10), (A7) and (A8):

$$e_j^u = (\Theta_j^{u\,\prime} / \Theta_j^u) / \{\Theta_j^{u\,\prime}\varepsilon_j^u(\varepsilon_j^u - 1)/(1 - \Theta_j^u\varepsilon_j^u(\varepsilon_j^u - 1)) + \Theta_j^{u\,\prime\prime} / \Theta_j^{u\,\prime}\} . \tag{A9}$$

This is an ordinary differential equation. To solve it I substitute $\Theta_j^u = \theta_j^u(\varphi_j^u)^{\lambda_j^u}$ where $\theta_j^u[>0]$ and $\lambda_j^u[>1]$ are constant. After some algebra, I get equation (11) of the text:

$$(\varphi_j^u)^{\lambda_j^u} = \{\lambda_j^u - e_j^u(\lambda_j^u - 1)\} / \{(\lambda_j^u + e_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u\}, \tag{A10}$$

Differentiating (A10) I get $\partial\varphi_j^u / \partial e_j^u < 0$; and since $\partial e_j^u / \partial r_j^u < 0$, I have $\partial\varphi_j^u / \partial r_j^u > 0$. Suppose $\varphi_j * \neq \text{Max}(\varphi_j^1, \cdots, \varphi_j^{n_j})$ and consider u such that $\varphi_j^u > \varphi_j *$. If investors increase r_j^u, V_j^u will go down, forcing u to increase φ_j^u. If r_j^u goes down, V_j^u will go up since, from (A6),

$$\partial V_j^u / \partial r_j^u = -p_j^u q_j^u * (\tilde{p}, t)\{(1 - \varepsilon_j^u(\varepsilon_j^u - 1)\Theta_j^u)/(r_j^u)^2\varepsilon_j^u + (\varepsilon_j^u - 1)\Theta_j^{u\,\prime}(\partial\varphi_j^u / \partial r_j^u)/r_j^u\} < 0,$$

and u has no reason to change φ_j^u. If $\varphi_j * = \text{Max}(\varphi_j^1, \cdots, \varphi_j^{n_j})$ and $\varphi_j^u < \varphi_j *$, V_j^u will decline as investors sell, r_j^u will increase and φ_j^u will go up to $\varphi_j *$. Thus, $\varphi_j^u * = \varphi_j^v * = \text{Max}(\varphi_j^1, \cdots, \varphi_j^{n_j})$. Since ϕ_j is same for all firms in industry j, relative costs do not change, so that, $\dot{\varepsilon}_j^u = \dot{\varepsilon}_j^v = 0$, $\Phi_j^u = \Phi_j^v = \Phi_j$, and $\rho_j^u = 0$ for all $u = 1, \cdots, n_j$, or, in the matrix form, with $[\varnothing]$ as null vector,

$$[\varepsilon - \text{I}][\Phi] = [\varnothing] . \tag{A11}$$

The set of solutions to (A11) is the null space of $[\varepsilon - \text{I}]$ which, from (5) comprises the unit vector and any of its multiples. Thus, with Φ_j as a number, the complete set of solutions for (A11) is

$$[\Phi] = \Phi_j \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} .$$

■

Proof of Proposition 2

For any integer λ_j^u, (A10) has λ_j^u roots given by

$\varphi_{jk}^u = \Lambda_j^u\{\cos(2k\pi/\lambda_j^u) + i\sin(2k\pi/\lambda_j^u)\}$,

where $k = 0,\cdots,(\lambda_j^u - 1)$; and Λ_j^u is a real root of $\{\lambda_j^u - e_j^u(\lambda_j^u - 1)\}/\{(\lambda_j^u + e_j^u)\varepsilon_j^u(\varepsilon_j^u - 1)\theta_j^u\}^{1/\lambda_j^u}$. Let $\upsilon(\text{"upsilon"}) \equiv \{\cos(2k\pi/\lambda_j^u) + i\sin(2k\pi/\lambda_j^u)\}[= 1^{1/\lambda_j^u}]$ for $k = 1$. The λ_j^u values of $1^{1/\lambda_j^u}[= 1, \upsilon, \upsilon^2, \cdots, \upsilon^{\lambda_j^u - 1}]$ form the vertices of a regular polygon of λ_j^u sides inscribed in a unit circle with one vertex at 1 (Figure 5). Each vertex of a polygon is a λ_j^uth root of unity. For $\lambda_j^u = 2$, I have two vertices with values 1 and -1 and no complex roots. For $\lambda_j^u = 4$, I have four vertices with values $1, i, -1$ and $-i$; real parts of both complex roots are zero. For no integer value of λ_j^u other than 2 and 4 will the real parts of complex roots of (A10) be non-negative. Extension of the argument to rational values yields one more feasible value of $\lambda_j^u[= 4/3]$ in which case there are four roots—two real and two complex with real parts of the complex roots zero. The proof can be extended to irrational values of λ_j^u by approximating them by rational values and taking limit.

Insert Figure 5 about here

Proof of Proposition 3

The question of stability trivially does not arise if $\varphi_j^u(r_j^u)[\equiv \xi_j^u(\tilde{p})/z_j^u] = 0$ for all u, j. I assume that $\varphi_j^u \neq 0$ for some u, j, and define $L(\tilde{p}) \equiv \sum_j \sum_u (\varphi_j^u - \varphi_j^{u*})^2/2$; $L(\tilde{p}) > 0$ if $\tilde{p} \neq \tilde{p}^*$ and $L(\tilde{p}) = 0$ if and only if $\tilde{p} = \tilde{p}^*$. Differentiating at $\tilde{p} \neq \tilde{p}^*$:

$\dot{L}(\tilde{p}) = \sum_j \sum_u (\varphi_j^u - \varphi_j^{u*})\dot{r}_j^u(\partial\varphi_j^u/\partial r_j^u)$.

From Proposition 1, $\varphi_j^{u*} = \text{Max}(\varphi_j^1, \cdots, \varphi_j^{n_j})$ and $V_j^u(\varphi_j^u) < V_j^u(\varphi_j^{u*})$. If $\varphi_j^u \neq \varphi_j^{u*}$ then $\varphi_j^u < \varphi_j^{u*}$, in which case investors will sell u's shares, its value will fall and r_j^u will go up so that $\dot{r}_j^u > 0$. Since $\partial\varphi_j^u/\partial r_j^u > 0$, $\dot{L}(\tilde{p}) < 0$. ∎

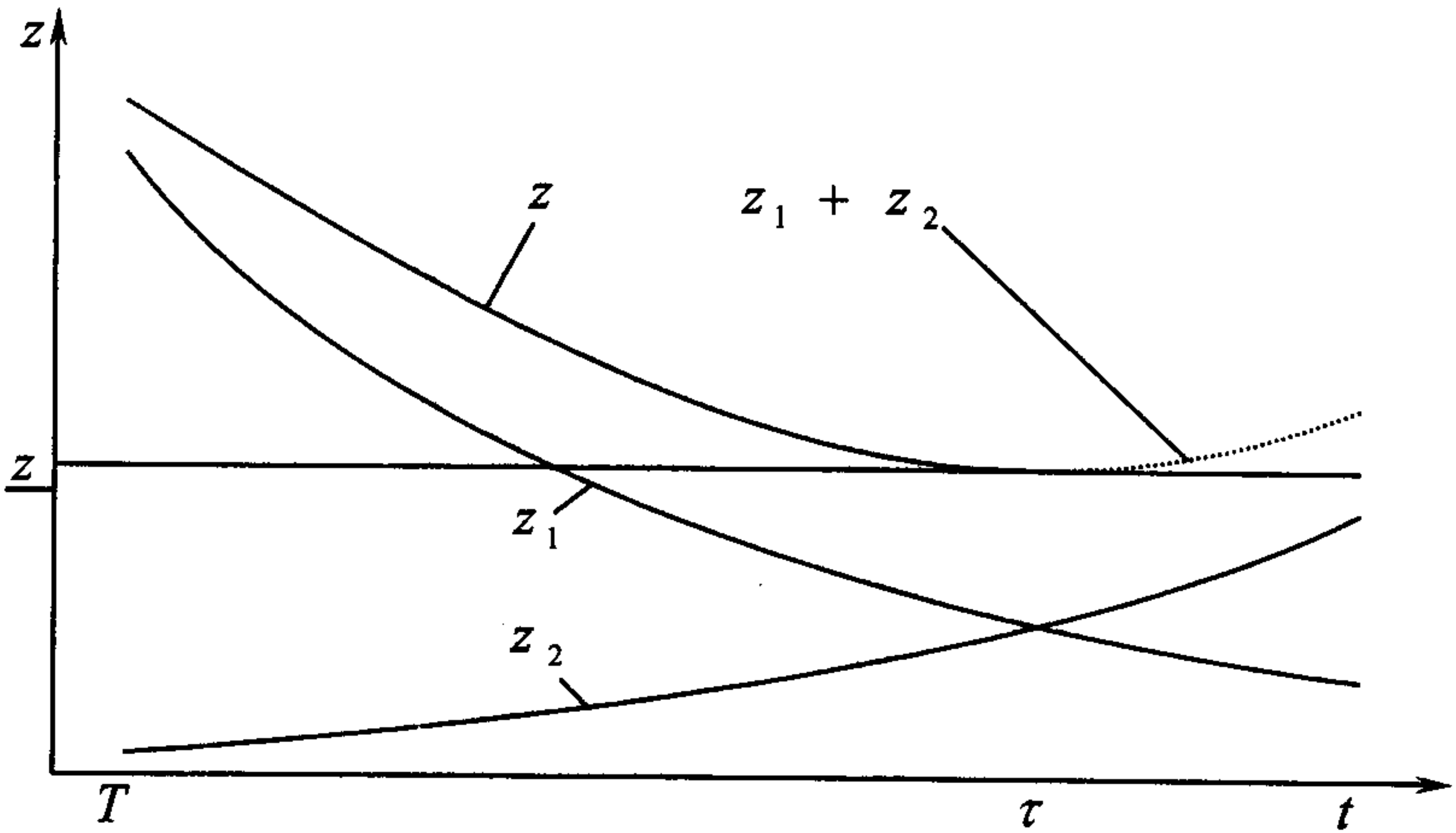

Figure 1(a): Time Paths of z and its components for $\lambda = 2$

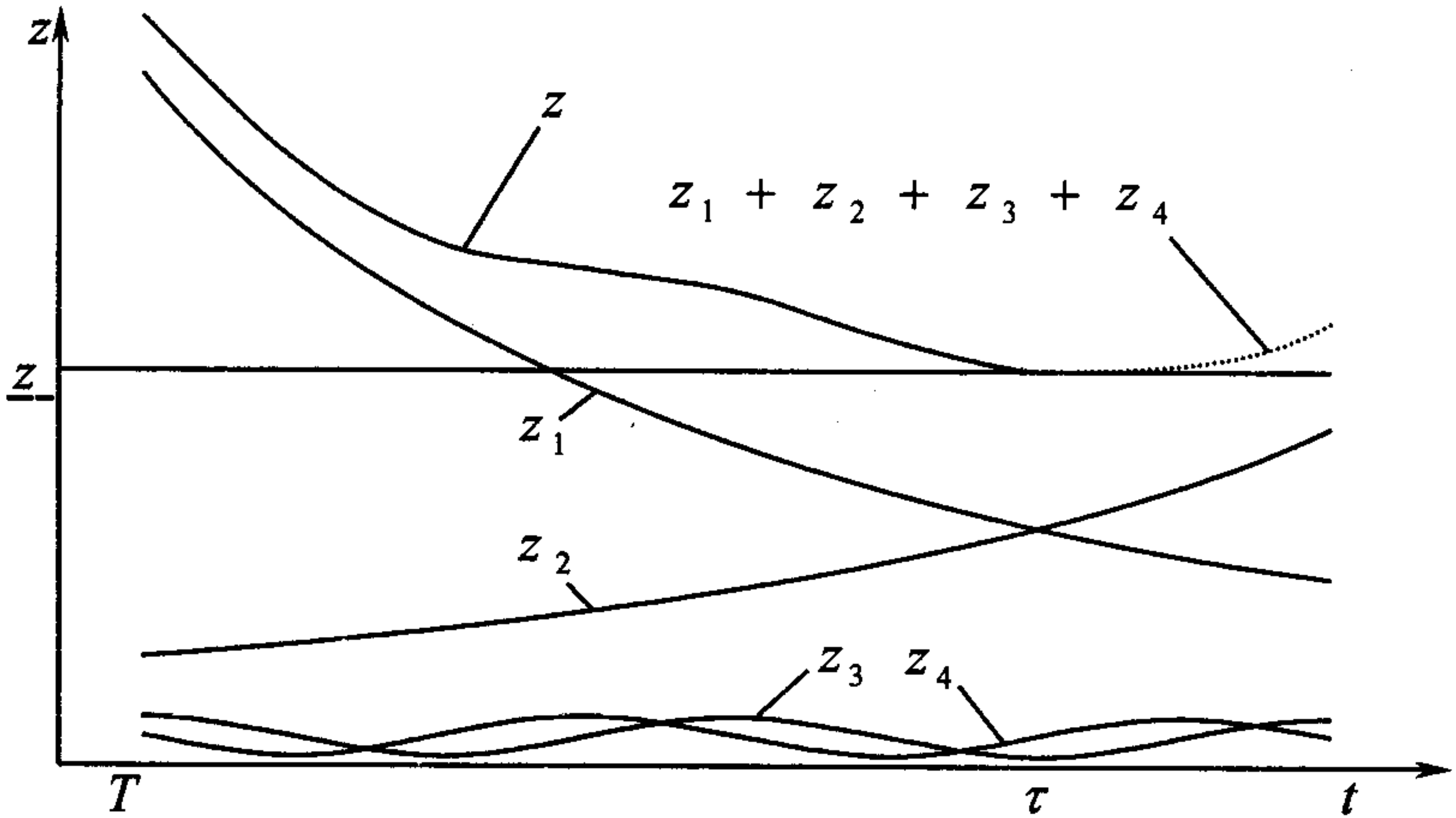

Figure 1(b): Time Paths of z and its components for $\lambda = 4$ and $\lambda = 4/3$

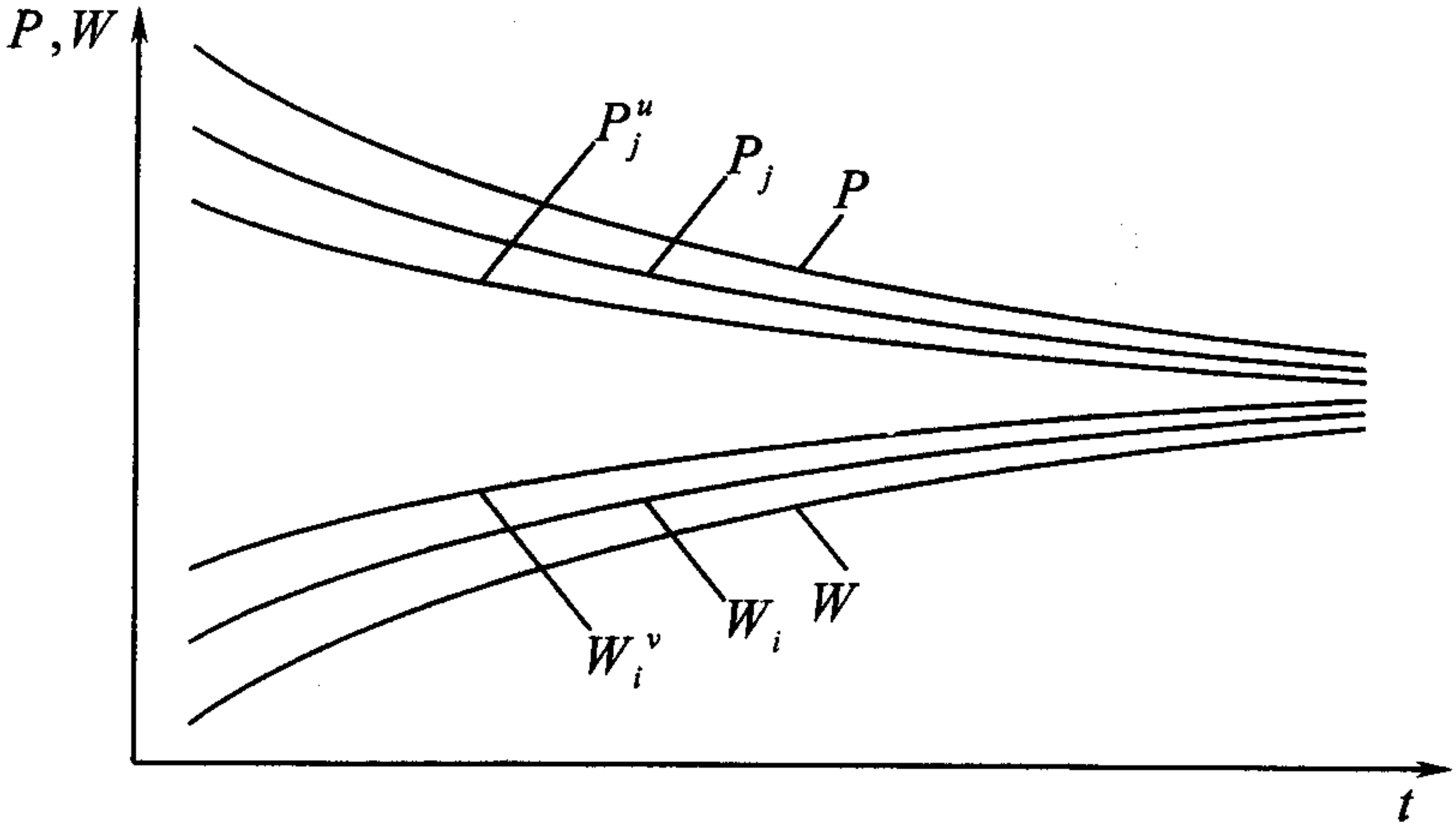

Figure 2(a): Time paths of real prices and wages for $\lambda = 2$

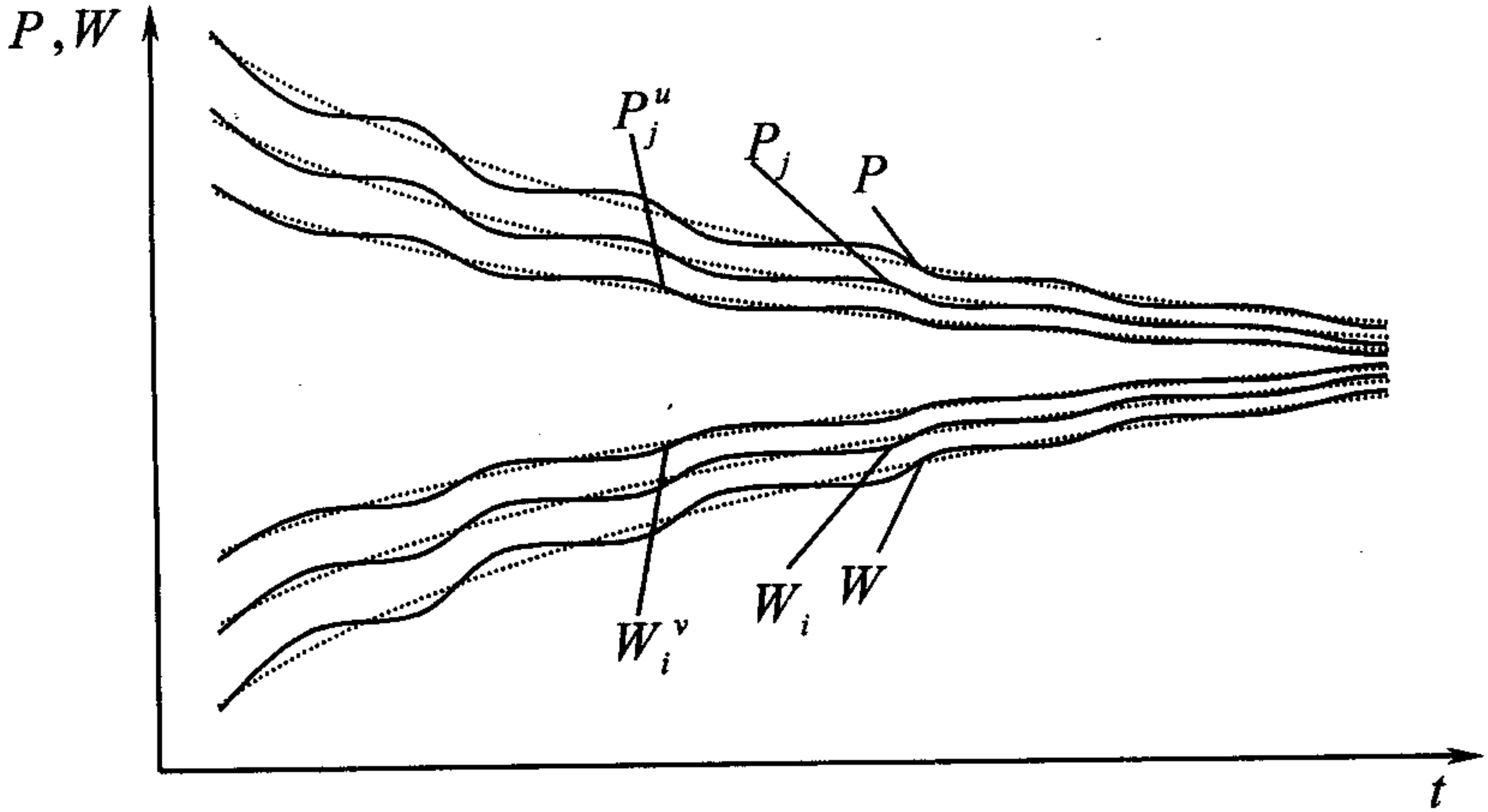

Figure 2(b): Time paths of real prices and wages for $\lambda = 4$ and $\lambda = 4/3$

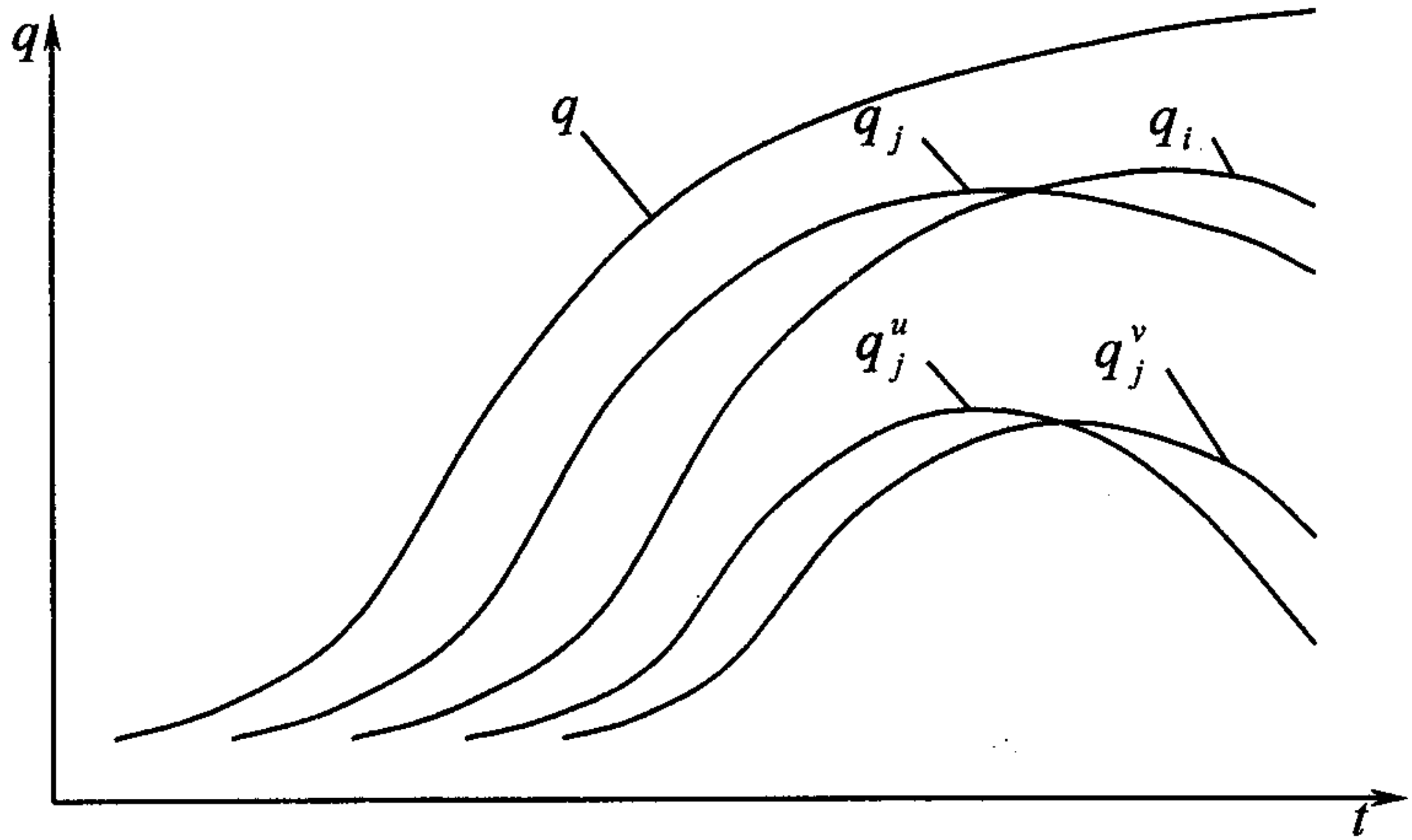

Figure 3(a): Time paths of output for $\lambda = 2$

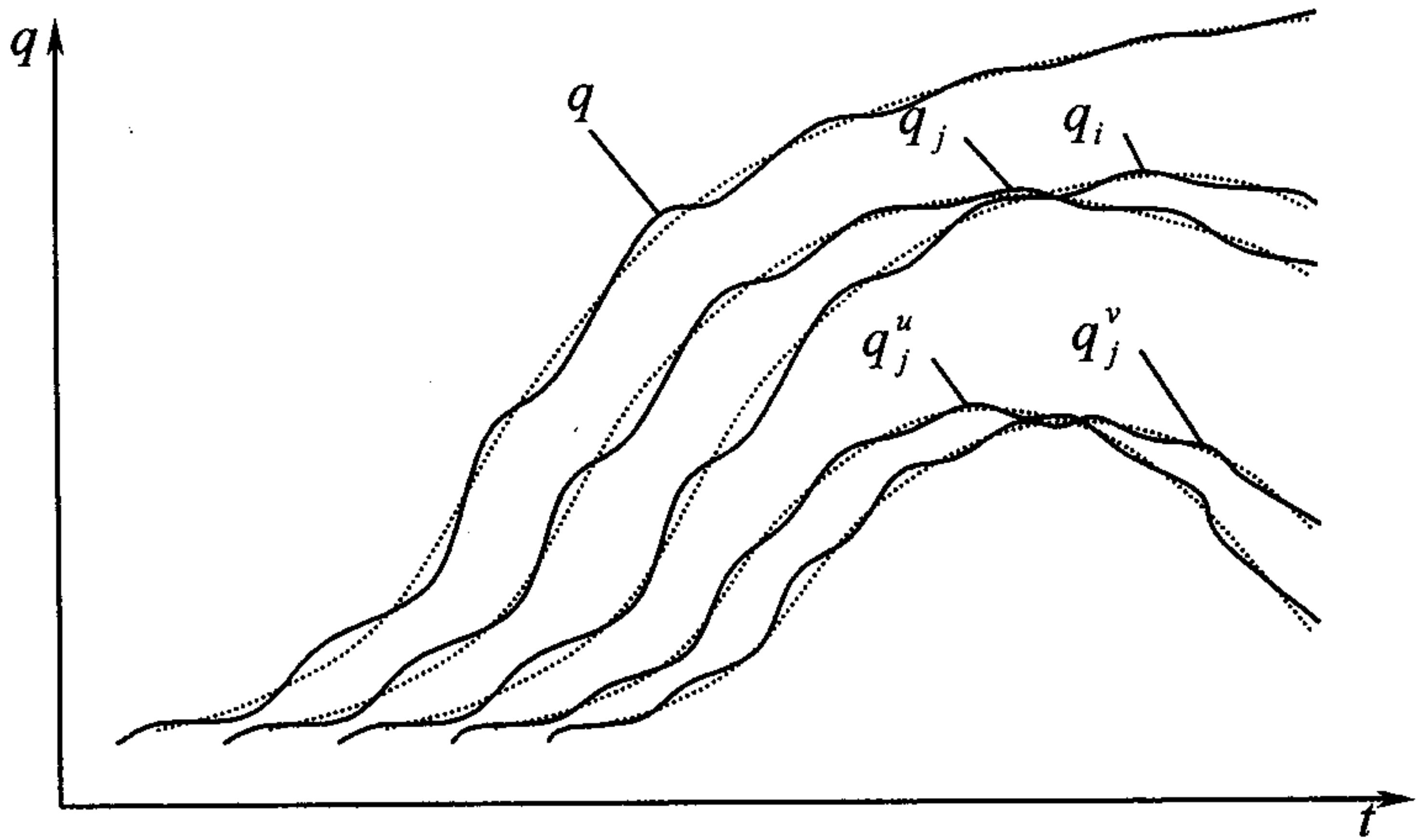

Figure 3(b): Time paths of output for $\lambda = 4$ and $\lambda = 4/3$

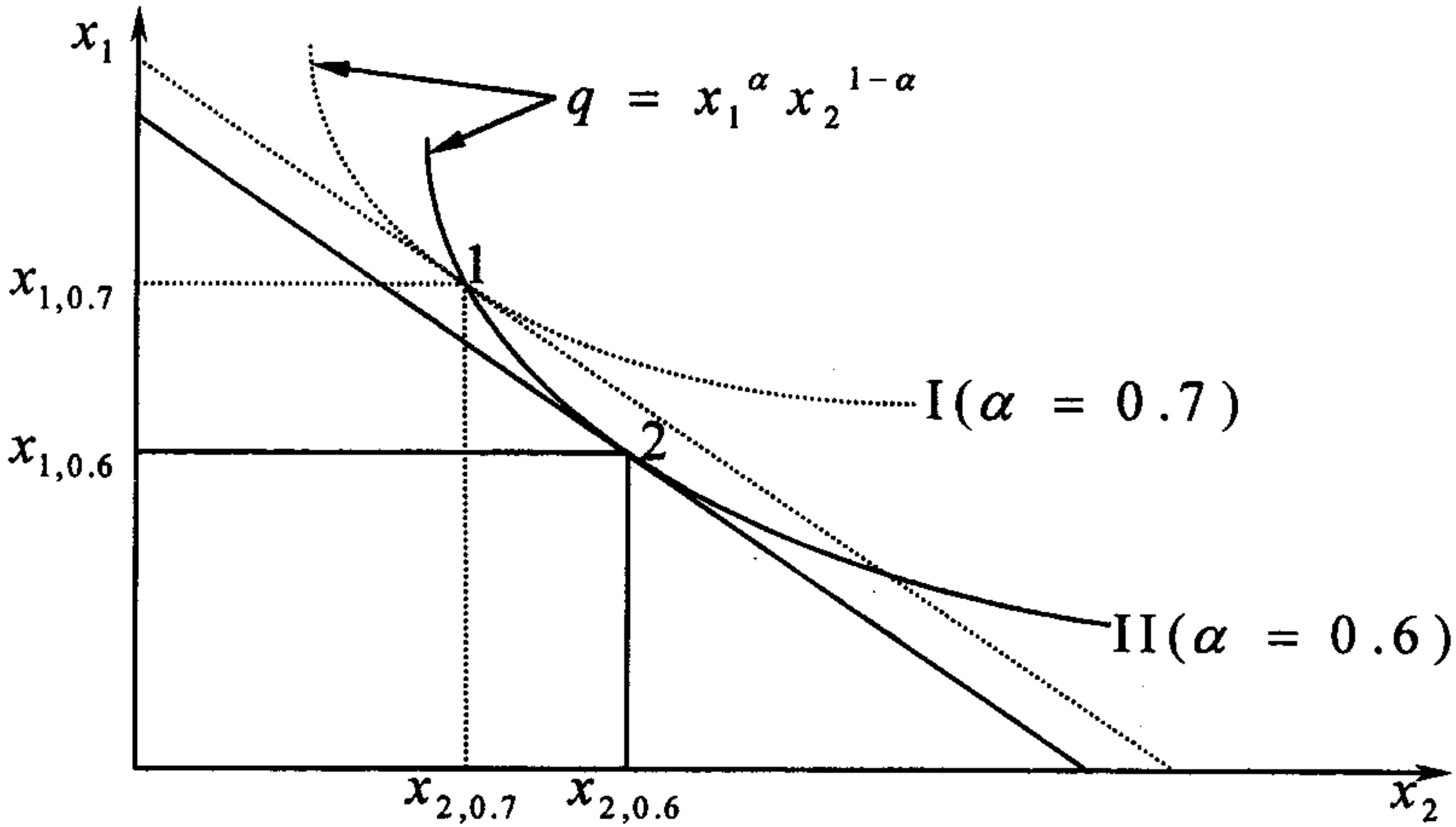

Figure 4(a): Technological change in the x-plane, $\lambda = 2$

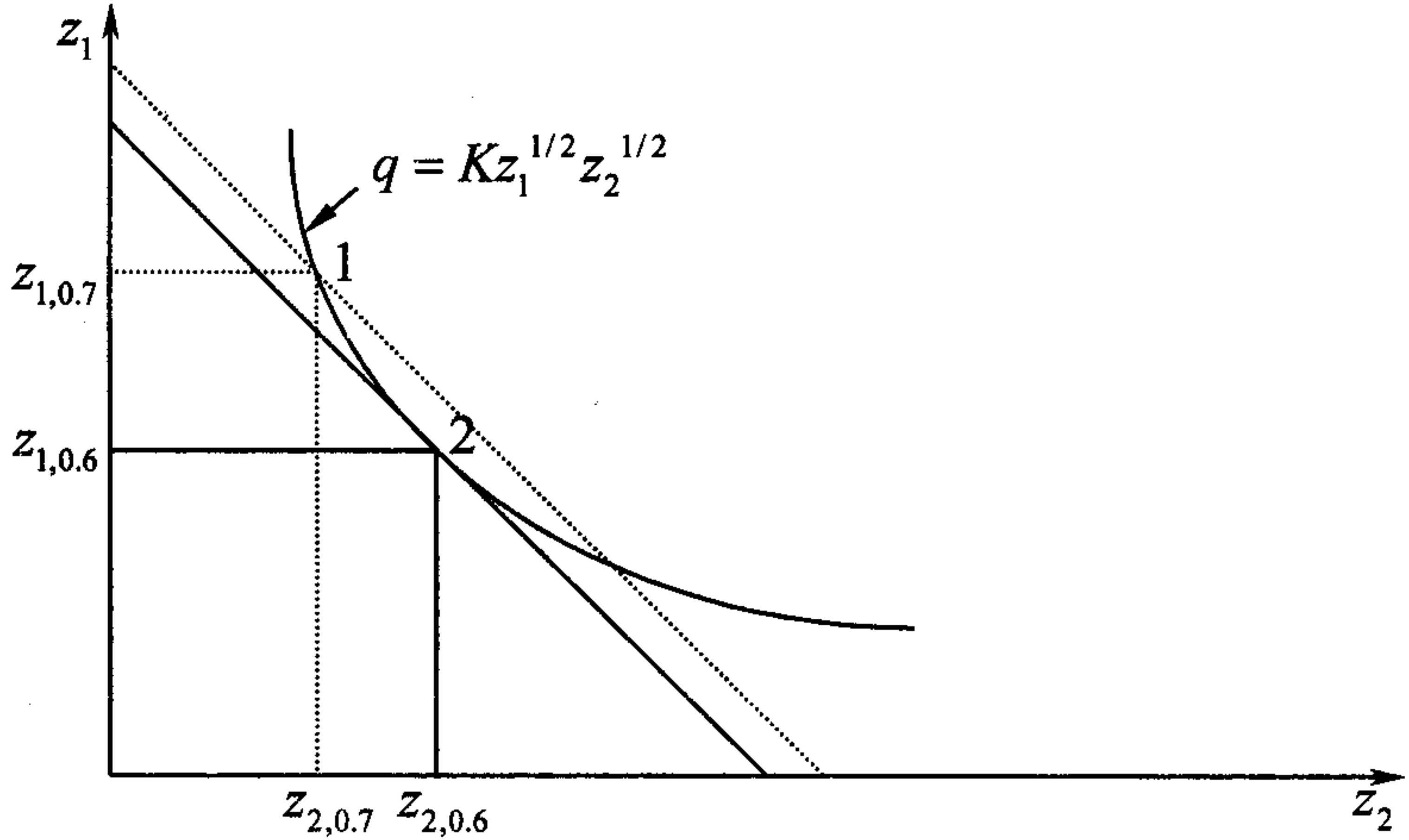

Figure 4(b): Technological change in the z-plane, $\lambda = 2$

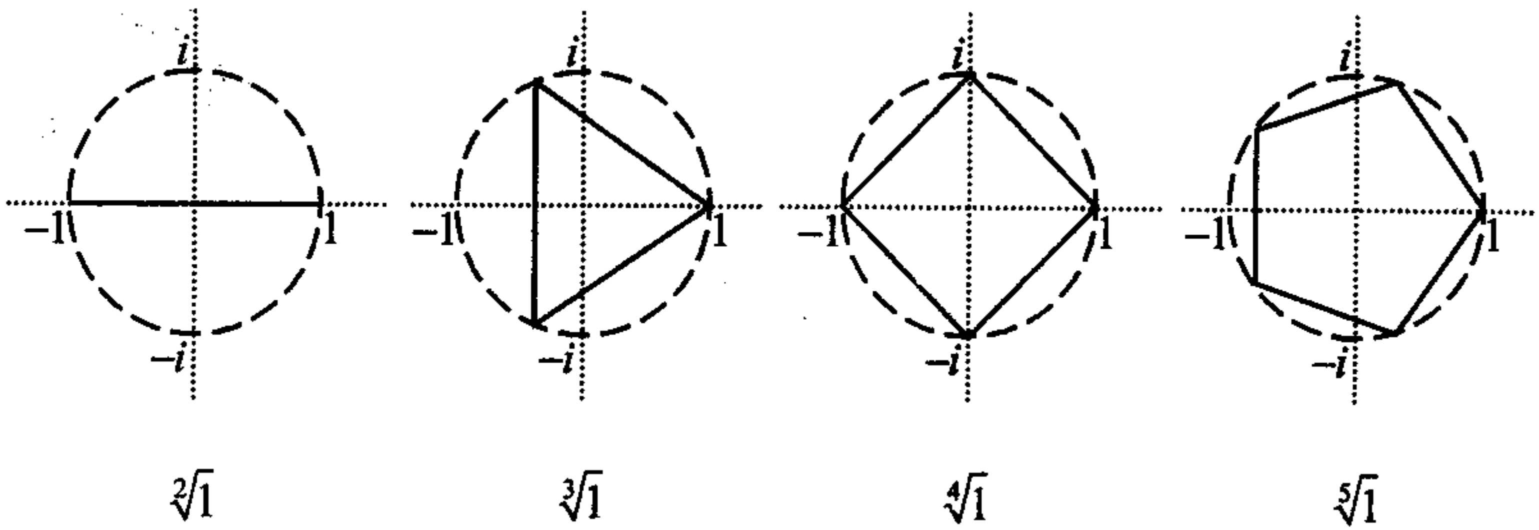

Figure 5: Roots of unity

	Old technology, $\alpha = 0.7$	New technology, $\alpha = 0.6$	
	$t < T$	$t = T$	$t = \tau$
x-plane $p_1 = \$200$; $p_2 = \$50$			
$q[= x_1^{\alpha} x_2^{1-\alpha}]$	1.000		1.000
$c[= (p_1/\alpha)^{\alpha}(p_2/(1-\alpha))^{1-\alpha}]$	\$243.060		\$225.160
x_1	0.851		0.680
x_2	1.458		1.801
z-plane Price of z_1 = price of z_2 = \$243.06 $K \approx 2.159$; $\bar{z} = 1$; $\underline{z}[= 225.16/243.06] = 0.926$			
$q[= K z_1^{1/2} z_2^{1/2}]$		1.000	1.000
$z[\equiv z_1 + z_2]$		1.000	0.926
$z_1[= (z + (z^2 - \underline{z}^2)^{1/2})/2]$		0.688	0.463
$z_2[= (z - (z^2 - \underline{z}^2)^{1/2})/2]$		0.312	0.463

Table 1: A numerical illustration of technological change with $\lambda = 2$ shown in Figure 4