# On stability of coalitions when externalities and stochasticity co-exist

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**Abstract**

We consider a class of cooperative games with transferable utilities where the payoff to a coalition is a function of the overall coalition structure (*externalities*) and the payoff to a coalition is not deterministic (*stochasticity*). Externalities and stochasticity in the cooperative game theory literature have almost always been studied separately. We propose a theoretical framework to analyze a situation when both are present together. We introduce a notion of stability and propose a related solution concept, called "foresighted nucleolus". We prove that the foresighted nucleolus always exists, but it may not be unique. We also provide a computational method and a numerical example to illustrate the solution concept.

*Keywords:* partition function games, stochastic payoffs, nucleolus

## 1 Introduction

Coalitional games with transferable utilities, represented in *characteristic function form*, is perhaps the most popular class of games studied in the literature. A restrictive assumption in such games is that the payoff to a coalition is independent of the coalition structure. However in many real situations, there exist *externalities* among coalitions, that is, the payoff to a coalition also depends on the non-members existing in the game. Thrall and Lucas (1963) propose *partition function*, which assigns to each coalition a payoff depending on the coalition structure. In the economic situations where the payoff of a coalition depends on other coalitions existing in the game, the partition function can be used to model the game. This class of games is called *partition function form games*.

A solution concept is developed as a set of prescriptions about how the payoff generated by a coalition should be distributed among its members. In cooperative game theory literature, there are many solution concepts, broadly classified into *fairness based concepts* and *stability based concepts*. In this paper we focus

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on stability based concept. A solution concept is said to be stable if the payoffs are distributed in such a manner that no coalition has an incentive to deviate. There are many solution concepts to analyze stability of characteristic function form games e.g. the core, the nucleolus, the bargaining set and the kernel. Some of the stability based solution concepts for partition function form games are—the equilibrium binding agreement (Ray and Vohra, 1997), $\gamma$-core (Chander and Tulkens, 1997), r-core (Huang and Sjostrom, 2003), the recursive core (Koczy, 2007), and the equivalence nucleolus (Tripathi and Amit, 2016).

Another restrictive assumption in the literature of characteristic function form games is that the payoff to a coalition is deterministic. This assumption is relaxed in a class of games called *stochastic cooperative games*. Stochastic cooperative games were introduced by Charnes and Granot (1973) as *chance-constrained games*. In chance-constrained games, characteristic function is a random variable. Charnes and Granot (1976, 1977) extend the classical solution concepts of characteristic function form games such as the core and the nucleolus to the chance-constrained games.

Coalitional game theory literature discusses the issues of externalities and stochasticity of payoffs under different classifications of games. However in many real situations both externalities and stochasticity of payoffs co-exist. In this paper, we consider a class of coalitional games with transferable utilities where the payoff to a coalition is a function of the coalition structure—*externalities*, and the payoff to a coalition in not deterministic—*stochasticity*. We develop a theoretical framework using partition function form games, but the partition function is defined to be a random variable. We define a solution concept, called *foresighted nucleolus* for stability of this class of games. Our idea behind the foresighted nucleolus is motivated by the equivalence nucleolus (Tripathi and Amit, 2016), which uses equality of dissatisfaction to characterize stability. We prove that the foresighted nucleolus always exists, but it may not be unique.

## 2 Foresighted Nucleolus

### 2.1 The model

**Definition 1** (Partition Function Form Games with Stochastic Payoffs)**.** *A partition function game with stochastic payoff is a pair $(N, \boldsymbol{U}(C,P))$, where $N$ is a finite set of players and $\boldsymbol{U}$ is a random variable assigning to each coalition $C$ in an embedded coalition $(C,P)$, a non-negative stochastic payoff $\boldsymbol{U}(C,P)$ conditional on partition $P$. $F_{\boldsymbol{U}(C,P)}$ denotes the cumulative probability distribution of the payoff to the coalition $C$ embedded in the partition $P$. Thus $F_{\boldsymbol{U}(C,P)}(t) = \boldsymbol{P}\{\boldsymbol{U}(C,P) \leq t\} \forall t \in \mathbb{R}$.*

A vector of deterministic payments $x = (x_1, x_2, \cdots, x_N)$ promised to the players of a game $\{N, \mathbf{U}(C,P)\}$ before the actual benefit is realized, is called *prior payoff vector*.

**Definition 2** (Probabilistic dissatisfaction)**.** *Probabilistic dissatisfaction of an embedded coalition $(C,P)$ conditional on partition $P$ with respect to a prior payoff vector $x$ is defined as the probability that $\boldsymbol{U}(C,P)$ is more than $x(C,P)$. It is denoted by $e_x(C,P)$ and given by $e_x(C,P) = 1 - F_{\boldsymbol{U}(C,P)}(x(C,P))$.*

**Definition 3** (Participation limit)**.** *Participation limit of coalition $C$, denoted by $\alpha_C$, is a probability value associated with the coalition, if the probabilistic dissatisfaction of $C$ does not exceed $\alpha_C$ for every partition $P$ in a game, that is, it should satisfy $1 - F_{\boldsymbol{U}(C,P)}(x(C,P)) \leq \alpha_C \ \forall P$.*

**Definition 4** (Payoff reasonability). *Payoff reasonability of prior payoff vector $x$ is a probability value associated with $x$ such that $1 - F_{\boldsymbol{U}(C,P)}(x(C,P)) \geq \beta_x, \forall C \in P$. It is denoted by $\beta_x$.*

**Definition 5** (Feasible payoff vector). *A set of prior payoffs $Y$ is called feasible with respect to partition $P$, if $Y = \{x; x_i \geq 0 (i = 1, 2, \ldots, n), \sum_{i \in C} x_i \geq F^{-1}_{\boldsymbol{U}(C,P)}(\beta_x), \forall C \in P\}$.*

**Definition 6** (Payoff configuration). *A payoff configuration to a game $(N, \boldsymbol{U}(C,P))$ is a pair $(P, x_P)$, where $P$ is a partition and $x_P$ is a feasible payoff vector with respect to $P$.*

## 2.2 The solution concept

For every player $i \in N$ in a game $(N, \mathbf{U}(C,P))$, *stochastic worth* of player $i$, denoted by $\mathbf{W}_i$, is a non-negative stochastic payoff which player $i$ assigns itself as a measure of its own valuation in the game. $F_{\mathbf{W}_i}$ denotes the cumulative distribution of the random variable $\mathbf{W}_i$. $x_i$ is the value which a player is promised in the prior payoff vector. Probability that $\mathbf{W}_i$ is more than $x_i$, denoted by $e_x(\{i\}, P)$, is the probabilistic dissatisfaction of player $i$. Foresighted nucleolus is the payoff configuration in which the probabilistic dissatisfaction of the most dissatisfied player is minimum.

**Assumption 1** (Stochastic dominance assumption). *Payoff to a coalition embedded in a more refined partition first order stochastically dominates the payoff to the same coalition embedded in a less refined partition.*

**Assumption 2** (Total order assumption). *Stochastic ordering of payoff distributions conditional on partitions, is always a total order.*

**Definition 7** (Stochastic worth). *Stochastic worth of a player $i$, $\boldsymbol{W}_i = \boldsymbol{U}(\{i\}, P'_{\{i\}}))$, where $P'_{\{i\}}$ is the coarsest residual partition of $P$ with respect to $i$, and $P$ is any partition of $N$.*

**Definition 8** (Equality of expected modified worth). *For any two players $i$ and $j$, which belong to the same coalition, the two random variables $(\boldsymbol{W}_i - x_i)$ and $(\boldsymbol{W}_j - x_j)$ should satisfy $\mathbb{E}(\boldsymbol{W}_i - x_i) = \mathbb{E}(\boldsymbol{W}_j - x_j)$, assuming that $(\boldsymbol{W}_i - x_i) \succeq_{st} (\boldsymbol{W}_j - x_j)$ is well defined.*

**Lemma 1.** *The payoff division rule of the equality of expected modified worth implies that $(\boldsymbol{W}_i - x_i) \approx_{st} (\boldsymbol{W}_j - x_j)$ $\forall i, j \in C$, where $C$ is a coalition in a given partition $P$.*

**Theorem 1.** *A payoff configuration $(P, x_P)$, where the prior payoff vector $x_P$ satisfies the constraints of—participation limit, payoff reasonability, non-negativity and equality of expected modified worth, always exists.*

**Theorem 2.** *For any partition $P$, a corresponding prior payoff vector $x_P$, which satisfies the constraints of participation limit, payoff reasonability, non-negativity and equality of expected modified worth, is not unique.*

**Definition 9** (Dissatisfaction sequence). *For a given payoff configuration $(P, x_P)$, the dissatisfaction sequence, denoted by $\delta^{\blacktriangleright}_{(P,x_P)}$, is a non-increasing sequence of probabilistic dissatisfaction values of all the players, that is, $\delta^{\blacktriangleright}_{(P,x_P)} = \langle e_{x_P}(\{i\}, P) \rangle, \forall i \in N$.*

**Definition 10** (Foresighted nucleolus). *Given a game $(N, \boldsymbol{U}(C,P))$, the foresighted nucleolus, denoted by $\boldsymbol{Nuc}(N, \boldsymbol{U}(C,P))$, is the payoff configuration $(P, x_P)$ which corresponds to the dissatisfaction sequence $\delta^*_{(P,x_P)}$ where $\delta^*_{(P,x_P)}$ is lexicographically minimal among all the possible dissatisfaction sequences of the game.*

**Theorem 3.** *Foresighted nucleolus always exists.*

**Theorem 4.** *Foresighted nucleolus may not be unique.*

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