Branching random walk in the presence of a hard wall

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Abstract

We consider the Branching Random Walk on d-ary tree of height n under the presence of a hard wall which restricts each value to be positive. The question of behavior of Gaussian process with long range interactions under the presence of a hard wall has been addressed, and a relation with the expected maxima of the processes has been established. We find the probability of event corresponding to the hard wall and show that the conditional expectation of the gaussian variable at a typical vertex, under positivity, is at least less than the expected maxima by order of $\log n$.

1 Introduction

Let us consider a d-ary tree of n levels and call it T_n. We define a Branching random walk on T_n and call it as $\{\phi_v^n : v \in T_n\}$. The covariance structure of this Gaussian process is given by the following

$$\begin{aligned} \operatorname{Var} \phi_v^n &= n \quad \text{for all } v \in T_n \\ \operatorname{Cov}(\phi_u^n, \phi_v^n) &= n - \frac{1}{2} d_T(u,v) \quad \text{for all } u \neq v \in T_n \,. \end{aligned} \tag{1}$$

where d_T denotes the tree distance.

We wish to find bounds on the order of the probability of a branching random walk being positive at all vertices. We also wish to find the expected value and size of a typical vertex under the condition that it is positive everywhere. The behavior that we are considering is that of entropic repulsion for these Gaussian fields which is it's behavior of drifting away when pressed against a hard wall so as to have enough room for local fluctuations, as is referred to in [15] . The phenomenon of entropic repulsion for gaussian free field has been studied in literature for some time now. The entropic repulsion for infinite Gaussian free field for dimension ≥ 3 has been studied in [3]. As a continuation to this, the finite field with zero boundary conditions for dimension ≥ 3 was studied in [9]. In case of the finite field, the positivity was looked at from two different angles, one involing the interior only, while the other considered the whole box. Both looked at the phenomenon of positivity of the field in a box of size N. Though the typical behavior of a vertex was similar, this order was not so, when positivity for the entire box was considered. But on removing positivity condition for a layer near the box, the order was same as in [3]. It was also shown in [9] that the probability of positivity in case of GFF in a box of dimension 2 decays exponentially, and this

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is really a boundary phenomenon. So in order to look into the long range correlations, and local fluctuations, the boundary effect has to removed. This approach has been taken in [2] to look into the behavior of a typical vertex when pressed against this hard wall for a Gaussian free field.

Works on GFF in a box of size n in dimension 2 since [2] have utilized the covariance of GFF in the interior of the box. It has been observed to be log-correlated. To further refine the results on entropic repulsion of the GFF in dimesion 2 it is imperative to consider a similar behavior for the BRW on a tree. The multi-scale analysis, hinting towards the tree structure is made use of extensively to study the extremal properties of GFF in dimesion 2. Similar strategies have been applied to study the entropic repulsion of Gaussian membrane model for the critical dimension 4 in [14]. It has been worked out in [16] that the Gaussian membrane model in the ciritical dimension is log-correlated. The works of [6], [4] and [13] further exhibit the strong relations between the BRW and log-correlated Gaussian fields, the branching number varying accoriding to the dimension of the box.

In the backdrop of that we study the behavior at a typical vertex of branching random walk on a d-ary tree. This is specially relevant keeping in mind the covariance structure of the BRW and that of the GFF in dimension 2, in the interior.

Entropic repulsion in case of GFF on Serpinski carpet graphs has been covered in [8]. More recently entropic repulsion in $|\nabla\phi|^p$ surfaces has been considered in [7].

We are interested in $\mathbb{P}(\phi^n_v \geq 0 \ \forall v \in T_n)$ as well as $\mathbb{E}(\phi^n_u \mid \phi^n_v \geq 0 \ \forall v \in T_n)$ and $\mathrm{Var}(\phi^n_u \mid \phi^n_v \geq 0 \ \forall v \in T_n)$.

As regards to the behavior of the branching random walk in presence of a hard wall, we recall similar results for other gaussian processes such as [10], [11], [2], [14], [8], [7]. The leading order term in the exponent of the probability of positivity is what is estimated, while we estimate both the leading order term and the second leading term in the exponent. This also helps us in finding the second order term in the expected value of a typical vertex, under the hard wall.

We know from [18] that $\mathbb{E}(\max_{v\in T_n} \phi_v)$ is of the form $c_1 n - c_2 \log n + O(1)$. Let us define $m_n = c_1 n - c_2 \log n$, and $\sigma^2_{d,n} = \frac{1-d^{-n}}{d-1}$.

Our main result of this paper, with regard to the probability of the hard wall, is the following:

Theorem 1.1. *There exists λ' such that $d^{c\lambda'}$ is of order n, such that for n sufficiently large we have, for $K_1, K_2, K_3 > 0$ independent of n,*

$$K_1 e^{-\frac{1}{2\sigma^2_{d,n}}(m_n-\lambda')^2 - K_3(m_n-\lambda')} \leq \mathbb{P}(\phi^n_v \geq 0 \ \forall v \in T_n) \leq K_2 e^{-\frac{1}{2\sigma^2_{d,n}}(m_n-\lambda')^2 - \frac{m_n-\lambda'}{c\sigma^2_{d,n}\log d}}. \tag{2}$$

In [2] it has been shown that the conditional expectation under positivity is roughly close to the expected maximum for the discrete GFF in 2 dimensions. Similarly in [14] a lower bound on the conditional expectation of a typical vertex, under positivity is computed to be close to the expected maxima. Here, however we show that for a branching random walk the conditional expectation is at least a constant times $\log n$ less than the expected maximum. The second main result of this paper is:

Theorem 1.2. *There exists positive numbers a, b, $a < b$ such that for all $v \in T_n$,*

$$m_n - b\log n + O(1) \leq \mathbb{E}(\phi^n_u \mid \phi^n_v \geq 0 \ \forall v \in T_n) \leq m_n - a\log n + O(1).$$

The approach that we take for proving this is that we raise the average value of the Gaussian process and then multiply a compensation probability to that. We optimize this average value so as to maximize the probabiltiy of positivity. The value at which this probability is maximized should ideally be the required conditional expectation.

In order to prove this in details, we invoke a new model called the switching sign branching random walk, which is similar in structure to the original branching random walk. We begin our calculations with a preliminary upper bound on the left tail of the maxima of the BRW in Section 2. Section 3 contains the definition of the new model switching sign branching random walk followed by a comparison of positivity for the branching random walk with this model using Slepian's lemma. A left tail computation for the maximum of this model gives us the ingrediants for proof of Theorem 1.1, which is in the concluding part of Section 4. Section 5 contains the proof of Theorem 1.2. The upper bound follows from Section 3, while for the lower bound we further have to invoke the Bayes' rule and tail estimates to arrive at our result.

Let us call the event $\{\phi_v^n \geq 0 \ \forall v \in T_n\}$ as Λ_n^+. First let us consider the sum of all the Gaussian variables at the level n and term it S_n. In mathematical terms $S_n = \sum_{v:v\in T_n} \phi_v^n$, where the sum contains d^n terms.

Remark 1.3. *The representation of the branching random walk, as a sum of two Gaussian fields, in the setting of entropic repulsion is a key point of the article. The constant part which represents the typical value of the field, helps in obtaining the height under the entropic replusion, while the covariance fluctuations remain restored in the other part. This representation helps in optimizing over the set of possible values for the typical height of the field under positivity.*

Remark 1.4. *Future directions along the line of this work include firstly the distributional behavior and convergence of the branching random walk under positivity. In [10] it has been shown that the infinite GFF for d $\geq$ 3 under positivity, on removing the conditioned height, converges weakly to the lattice free field. Whether a similar phenomenon can be observed in case of BRW is something that can be considered.*

Remark 1.5. *Furthering our work, we can also consider the similar phenomenon for general log-correlated Gaussian fields. The splitting of the covariance matrix into two parts, one involving a constant Gaussian field, is not immediate in case of log-correlated Gaussian fields as in the form considered in [13].*

2 Left tail of maximum of BRW

This section is dedicated to proving an exponential upper bound on the left of the maxima of a BRW. We first begin with a comparison lemma by Slepian for Gaussian processes in [17].

Lemma 2.1. *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a : a \in \mathcal{A}\}$ and $\{Y_a : a \in \mathcal{A}\}$ be two centered Gaussian processes such that: $\mathbb{E}(X_a - X_b)^2 \geq \mathbb{E}(Y_a - Y_b)^2$, for all $a, b \in \mathcal{A}$ and $\mathrm{Var}(X_a) = \mathrm{Var}(Y_a)$ for all $a \in \mathcal{A}$. Then $\mathbb{P}(\max_{a\in\mathcal{A}} X_a \geq \lambda) \geq \mathbb{P}(\max_{a\in\mathcal{A}} Y_a \geq \lambda)$ for all $\lambda \in \mathbb{R}$.*

The main result of the section is the following:

Lemma 2.2. *There exists constants $\bar{C}, c^* > 0$ such that for all $n \in \mathbb{N}$ and $0 \leq \lambda \leq (n)^{2/3}$,*

$$\mathbb{P}(\max_{v\in T_n} \phi_v^n \leqslant m_n - \lambda) \leq \bar{C}e^{-c^*\lambda} \tag{3}$$

Proof. From [18, Section 2.5] we have tightness for $\{\max_{v\in T_n}\phi_v^n - m_n\}_{n\in\mathbb{N}}$. So there exists $\beta > 0$ such that for all $n \geq 2$,

$$\mathbb{P}(\max_{v\in T_n}\phi_v^n \geqslant m_n - \beta) \geqslant 1/2. \tag{4}$$

Further, we also have that for some $\kappa > 0$ and for all $n \geq n' \geq 2$

$$\sqrt{2\log d}(n-n') - \frac{3}{2\sqrt{2\log d}}\log(n/n') - \kappa \leqslant m_n - m_{n'} \leqslant \sqrt{2\log d}(n-n') + \kappa. \tag{5}$$

Now let us fix $\lambda' = \lambda/2$ and $n' = n - \frac{1}{\sqrt{2\log d}}(\lambda' - \beta - \kappa - 4)$. From (5) it follows then that $m_n - m_{n'} \leqslant \lambda' - \beta$. Consider a tree of height n and look at its subtrees at height $n-n'$, which are individually trees of height n'. The total number of subtrees we have is $d^{n-n'}$. Let us call them $\{T_{n'}^{(1)}, T_{n'}^{(2)}, \ldots, T_{n'}^{(d^{n-n'})}\}$. Now for all $v \in T_n$, we define

$$\bar{\phi}_v^n = g_v^{n'} + \phi,$$

where g_v^n are the BRWs obtained by adding the Gaussians for the edges only in the subtrees of height n', and ϕ is a Gaussian of mean 0 and variance $n-n'$. Clearly

$$\operatorname{Var}\phi_v^n = \operatorname{Var}\bar{\phi}_v^n \quad \text{and} \quad \mathbb{E}\phi_v^n\phi_u^n \leq \mathbb{E}\bar{\phi}_v^n\bar{\phi}_u^n \ \forall u \neq v \in T_n.$$

So by Lemma 2.1, we have

$$\mathbb{P}(\max_{v\in T_n}\phi_v^n \leq t) \leq \mathbb{P}(\max_{v\in T_n}\bar{\phi}_v^n \leq t) \ \forall\, t \in \mathbb{R}. \tag{6}$$

Using (4) and (5), one has for all $i \in \{1, 2, \ldots, d^{n-n'}\}$,

$$\begin{aligned}\mathbb{P}(\sup_{v\in T_{n'}^{(i)}} g_v^{n'} \geq m_n - \lambda') &= \mathbb{P}(\sup_{v\in T_{n'}^{(i)}} g_v^{n'} \geq m_{n'} + m_n - m_{n'} - \lambda') \\ &\geq \mathbb{P}(\sup_{v\in T_{n'}^{(i)}} g_v^{n'} \geq m_{n'} - \beta) \geq 1/2\end{aligned}$$

and so $\mathbb{P}(\sup_{v\in T_n} g_v^{n'} < m_n - \lambda') \leq (\frac{1}{2})^{d^{n-n'}}$.

Therefore,

$$\mathbb{P}(\sup_{v\in T_n}\bar{\phi}_v^n \leq m_n - \lambda) \leq \mathbb{P}(\sup_{v\in T_n} g_v^{n'} < m_n - \lambda') + \mathbb{P}(\phi \leq -\lambda') \leq \bar{C}e^{-c^*\lambda},$$

for some $\bar{C}, c^* > 0$. Now in conjunction with (6), the lemma is proved. ☐

3 Switching Sign Branching Random Walk

At this juncture we start defining a new Gaussian process on the tree, which we call the switching sign branching random walk. This was used to approximate the branching random walk in [12] in case of a 4-ary tree. We have generalized the process for a d-ary tree. The switching sign branching random walk consists of two parts, one that varies across vertices, and the other that is fixed over vertices. The first part of the process, which is not fixed over vertices, is different from the normal branching random walk in the sense that instead of the d-edges coming out of it being associated to independent normal random variables, they are associated to linear combinations of $d-1$ independent Gaussians, such that the covariance between any two of them is the same, and all of them add up to zero. The existence of this is guaranteed by the following Lemma.

Lemma 3.1. *There exists $A \in \mathbb{R}^{(d-1)\times(d-1)}$ such that for $X \sim N(0, \sigma^2 I_{(d-1)\times(d-1)})$, the covariance matrix of AX has all its diagonal entries to be σ^2 and all its off-diagonal entries to be equal(say b). Further $\mathrm{Var}(1^T AX) = \sigma^2$ and $\mathrm{Cov}(-1^T AX, (AX)_i) = b$ for all $i \in \{1, 2, \ldots, d-1\}$.*

Proof. We know that the covariance matrix for AX is AA^T. Further from the condition that $\mathrm{Var}(1^T AX) = \sigma^2$ we get that $b = -\frac{\sigma^2}{d-1}$. So in order for A to exist we must have

$$AA^T = \sigma^2 \begin{bmatrix} 1 & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\ -\frac{1}{d-1} & 1 & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{d-1} & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & 1 \end{bmatrix}_{(d-1)\times(d-1)} .$$

Since the matrix on the right hand side is a symmetric matrix with non-negative eigenvalues, so by Cholesky decomposition we obtain the existence of such an A. ☐

A pictorial representation of a node for this process is given in Figure 1.

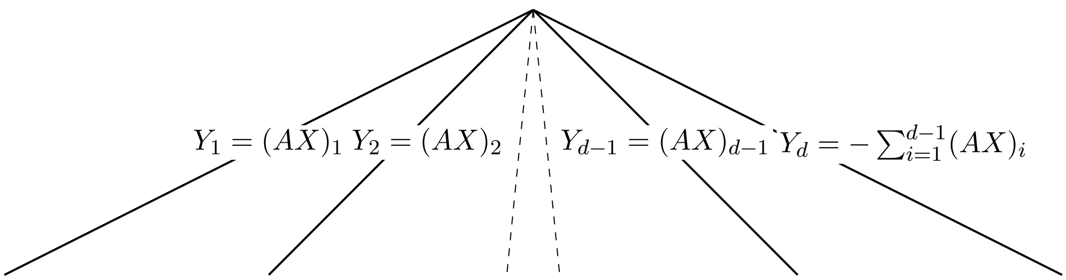

Figure 1: Node of the varying part of SSBRW

Now in the actual construction, unlike the BRW, we use a different value for σ^2 for each level l such that $1 \leq l \leq n$. Here level 1 denotes the edge connecting the root to its children and level n denotes the edges joining the leaf nodes to their parents. Let us denote this switching sign branching random walk on the tree T_n as $\{\xi_v^n : v \in T_n\}$. For $v \in T_n$ let us denote the Gaussian variable that is added on level l, on the path connecting v to the root, by $\phi_v^{n,l}$. Then we assign $\mathrm{Var}(\phi_v^{n,l}) = 1 - d^{-(n-l+1)}$. The switching sign branching random walk will consist of two parts, the first coming from the contribution at different levels in the tree which we call $\tilde{\phi}_v^n \stackrel{def}{=} \sum_{l=1}^n \phi_v^{n,l}$.

Finally we define the switching sign branching random walk as

$$\xi_v^n = \tilde{\phi}_v^n + X \tag{7}$$

where X is a Gaussian variable with mean zero and variance $\frac{1-d^{-n}}{d-1}$.

The covariance structure for this new model closely resembles that of the branching random walk. The following lemma deals with this comparison:

Lemma 3.2. *The Gaussian fields $\{\xi_v^n : v \in T_n\}$ and $\{\phi_v^n : v \in T_n\}$ are identically distributed.*

Proof. First we show that the variances are identical for the two processes. To this end,

$$\begin{aligned}\mathrm{Var}(\xi_v^n) &= 1-d^{-1}+1-d^{-2}+\cdots+1-d^{-n}+\frac{1-d^{-n}}{d-1}\\ &= n-\frac{1-d^{-n}}{d-1}+\frac{1-d^{-n}}{d-1}=n.\end{aligned}$$

Next in case of the covariances suppose we consider $u, v \in T_n$, such that they are separated until level k i.e $\mathrm{Cov}(\phi_u^n, \phi_v^n) = n-k$. Then we have

$$\mathrm{Cov}(\tilde{\phi}_u^n, \tilde{\phi}_v^n) = -\frac{1-d^{-k}}{d-1}+\sum_{l=k+1}^{n}(1-d^{-l}) = n-k-\frac{1-d^{-n}}{d-1}.$$

So, the covariance structures for the fields ξ and ϕ match, and hence they are identically distributed. □

A simple corollary of Lemma 3.2, is the following, based on the fact that the two processes have identical distributions.

Corollary 3.3. *We have the following equality:*

$$\mathbb{P}(\phi_v^n \geq 0 \;\forall v \in T_n) = \mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X) \tag{8}$$

Corollary 3.4. *From [18, Theorem 4], we have* $\mathbb{E}\max_{v\in T_n}\phi_v^n = n\sqrt{2\log d} - \frac{3}{2\sqrt{2\log d}}\log n + O(1)$. *Therefore,*

$$\mathbb{E}\max_{v\in T_n}\tilde{\phi}_v^n = n\sqrt{2\log d}-\frac{3\log n}{2\sqrt{2\log d}}+O(1).$$

Corollary 3.5. *There exists constants* $\bar{C}', c^* > 0$ *such that for all* $n \in \mathbb{N}$ *and* $0 \leq \lambda \leq (n)^{2/3}$,

$$\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leqslant m_n-\lambda) \leq \bar{C}'e^{-c^*\lambda} \tag{9}$$

Proof.

$$\frac{1}{2}\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leqslant m_n-\lambda) = \mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leqslant m_n-\lambda, X\leq 0) \leq \mathbb{P}(\max_{v\in T_n}\phi_v^n \leqslant m_n-\lambda).$$

Now using (3), and with $\bar{C}' = 2\bar{C}$ we arrive at (9). □

4 Estimates on left tail and positivity

From the (8) we understand that the probability of positivity for the branching random walk can be computed using bounds on the left tail of the maximum of $\tilde{\phi}_\cdot^n$, a part of the switching sign branching random walk, as the left tail is heavily concentrated around the maximum. This motivates the following computations on the left tail of the maximum.

Lemma 4.1. *Let us call* $c = 1/c_1$ *(where* $m_n = c_1 n - c_2\log n$*) to be the constant such that* $|m_{n-c\lambda} - m_n - \lambda| \to 0$ *as* $n\to\infty$*, where* λ *is of lower order than* n*. Then there exists independent constants* C', C'', K', K'' *such that for sufficiently large* n *we have*

$$K'\exp(-K''d^{c\lambda}) \leq \mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \leq m_n-\lambda) \leq C'\exp(-C''d^{c\lambda}). \tag{10}$$

Proof. We work with $\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda})$ as due to our definition of c, for sufficiently large n this probability is close to $\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_n-\lambda)$. This comes from the fact that, from [18], $\{\max_{v\in T^n}\tilde{\phi}_v - m_n\}$ converges in distribution, and so by an application of Slutsky's theorem $\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda})$ and $\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_n-\lambda)$ converge to the same value. We know that the SSBRW is a Gaussian field which is obtained by adding the same Gaussian to all vertices of a BRW. This helps us find bounds on lower and upper tails of maxima using results on convergence of maxima of BRW, as proved in [?], [5] etc.

We first consider the tree only up to the level $c\lambda$ and consider the cumulative sum of the Gaussian variables at these vertices till the level $c\lambda$. Let us rename all these Gaussian variables at level $c\lambda$ of this new tree to be $A_1, A_2, \ldots, A_{d^{c\lambda}}$. We know that the definition in Section 3 of switching sign branching random walk model guarantees $\sum_{i=1}^{d^{c\lambda}} A_i = 0$. Let us consider the subtrees rooted at the vertex which has values A_i and call its maximum to be M_i. These are trees of height $n-c\lambda$ and hence we have $\mathbb{E}M_i = m_{n-c\lambda}+O(1)\ \ \forall i$ and $M := \max_{v\in T^n}\tilde{\phi}_v = \max_{i=1}^{d^{c\lambda}}(M_i+A_i)$. We want to obtain bounds for the probability $\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda})$. We condition on the values of $A_1, A_2, \ldots, A_{d^{c\lambda}}$ which in turn breaks down the required probability in a product form since the maxima for the $d^{c\lambda}$ subtrees are independent and have identical distributions. We consider two different cases:

1) When $A_i^- \le 2\bar{A}$ for at least $d^{c\lambda}/2$ many i, where $\bar{A}$ is a positive constant to be chosen later on.

2) When 1) doesn't happen and so then $\sum_{i=1}^{d^{c\lambda}} A_i^- \ge \bar{A}d^{c\lambda}$.

For the first case we break it down into two parts according to when $\sum_{i=1}^{d^{c\lambda}} A_i^- \ge \bar{A}d^{c\lambda}$ or not. Now we have

$$\begin{aligned}
& \mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d^{c\lambda}}) \\
= \;& \mathbb{P}(\max_{i=1}^{d^{c\lambda}}(M_i+A_i) \le m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d^{c\lambda}}) \\
= \;& \prod_{i=1}^{d^{c\lambda}} \mathbb{P}(M_i + A_i \le m_{n-c\lambda} \mid A_i) \qquad < \text{from independence} > \\
\le \;& \prod_{i:A_i>0}^{d^{c\lambda}} \mathbb{P}(M_i \le m_{n-c\lambda} - A_i \mid A_i) \\
\le \;& \bar{C'}^{d^{c\lambda}} \exp(-c^* \sum_{i=1}^{d^{c\lambda}} A_i^+) = \exp(d^{c\lambda}\log\bar{C'} - c^*\sum_{i=1}^{d^{c\lambda}} A_i^-)
\end{aligned}$$

In the final two steps we first make use of (9), followed by the fact that $\sum_i A_i = 0$. For the cases where $A_i < 0$ we bound the terms in the product by 1. When 2) holds then clearly this is bounded by $\exp(-(c^*\bar{A} - \log\bar{C'})d^{c\lambda})$ and now on choosing $\bar{A}$ such that $(c^*\bar{A} - \log\bar{C'}) > 0$ we have $c^{**} > 0$ such that our required term is bounded by $\exp(-c^{**}d^{c\lambda})$. In the other case also

$$\mathbb{P}(M_i \le m_{n-c\lambda} - A_i \mid A_i) \le \mathbb{P}(M_i \le m_{n-c\lambda} + 2\bar{A})$$

for those i for which $A_i^- \le 2\bar{A}$. From lower bound on right tail of maximum, we can find p, independent of n, where $0 < p < 1$ such that $\mathbb{P}(M_i \le m_{n-c\lambda} + 2\bar{A}) < p$ for all sufficiently large n and so the probability is bounded by $\exp(-\bar{c}d^{c\lambda})$. Now from this $\bar{c}$ and c^{**} we select one unified C', C'' so that

$$\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda}) \le C'\exp(-C''d^{c\lambda}).$$

Again for the lower bound we have

$$\begin{aligned}\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda}) &= \int_{\mathbb{R}^{d^{c\lambda}}}\prod_{i=1}^{2^{c\lambda}}\mathbb{P}(M_i \le m_{n-c\lambda}-A_i)dA_i \\ &\ge (\bar{p})^{d^{c\lambda}}\int_{[-1,1]^{d^{c\lambda}}}\prod_{i=1}^{d^{c\lambda}} dA_i\end{aligned}$$

where $\bar{p}$ is chosen to be a lower bound on $\mathbb{P}(M_i \le m_{n-c\lambda}-1)$ for all sufficiently large n, which can be obtained from using convergence results on maxima of branching random walk. Now $\{A_1, A_2, \ldots, A_{d^{c\lambda}}\}$ are obtained by linear combinations of $d^{c\lambda}-1$ independent standard normal random variables, each being obtained from $c\lambda$ many of them, and a way to make all A_i's in the range $[-1,1]$ is to make absolute value of the contribution at the jth level to be bounded by $\frac{1}{10(c\lambda+1-j)^2}$, for $j=1,2,\ldots,c\lambda$. So the independent standard normals at level j are bounded by $\frac{1}{10\sqrt{d}(c\lambda+1-j)^2}$. So this gives, for some constant $K>0$,

$$\mathbb{P}(\max_{v\in T^n}\tilde{\phi}_v \le m_{n-c\lambda}) \ge (\bar{p})^{d^{c\lambda}}\prod_{j=1}^{c\lambda}\left(\frac{1}{10K\sqrt{d}(c\lambda+1-j)^2}\right)^{(d-1)d^{j-1}}.$$

Approximation of the sum, as shown below in Lemma 4.2 proves (10). □

Lemma 4.2. *$\sum_{j=1}^{n}(\log|n+1-j|)d^j$ is of order d^n.*

Proof. We begin with an upper bound on the sum. We use a trivial bound of $\log|x| \le |x|$ for $|x|\ge 1$, followed by a few series summations.

$$\begin{aligned}\sum_{j=1}^{n}(\log|n+1-j|)d^j &\le \sum_{j=1}^{n}(|n+1-j|)d^j \\ &= (n+1)\sum_{j=1}^{n}d^j - \sum_{j=1}^{n}jd^j \\ &= (n+1)\frac{d^{n+1}-d}{d-1} - \frac{nd^{n+2}-(n+1)d^{n+1}+d}{(d-1)^2} \\ &= \frac{d^{n+2}-(n+1)d^2+nd}{(d-1)^2}\end{aligned}$$

This gives the upper bound to be of order d^n. The lower bound follows easily. □

We now look back into our question of the branching random walk being positive at all vertices. We know that the maximum of the BRW is heavily concentrated around the expected maximum. Using this fact, in a neighborhood around the maximum, we further try to maximize the probability of the maximum being there. This point where this occurs will also roughly be the typical value of a vertex. This motivates the proof of Theorem 1.1.

Proof. of theorem 1.1 *Upper bound:* From (8) we have an upper-bound on the probability of positivity based on the switching signs branching random walk. We optimize this bound by first raising the mean to a level and look at the compensation we have to apply correspondingly. We optimize over these two to obtain our bound. We apply a similar strategy for obtaining the lower bound as well. Let us recall (8) at this juncture along with X, and let us call the variance of X to be $\sigma_{d,n}^2 = \frac{1-d^{-n}}{d-1}$. In (8), we condition on the value of X to obtain the following:

$$\mathbb{P}(\Lambda_n^+) = \frac{1}{\sigma_{d,n}\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x) \exp(-x^2/2\sigma_{d,n}^2) dx$$

Now, since the left tail of the maximum of a log-correlated Gaussian field, is heavily concentrated. So we may as well replace x by $m_n - \lambda$, and then integrate over λ. We split the integral into three parts, first with $\{-\infty < \lambda \leq 0\}$, second with $\{\frac{3}{c}\log_d n \leq \lambda < \infty\}$ and the rest. From tail estimates of a Gaussian, the first part is bounded by $O(\exp(-\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda')^2))$. From (10), we know that the second part is bounded by $C'\exp(-C''n^3)$. The rest part has an upper bound:

$$\frac{C'}{\sqrt{2\pi}} \int_0^{\frac{3}{c}\log_d n} \exp(-C''d^{c\lambda}) \exp(-(m_n - \lambda)^2/2) d\lambda. \tag{11}$$

We maximize the integrand in (11), over the range of the integral, to obtain an optimal λ, say λ', which is of order $\log n$. It satisfies the equation

$$m_n - \lambda' = \sigma_{d,n}^2 C'' c d^{c\lambda'} \log d.$$

Plugging in we obtain an upper bound as in (2).

Lower bound: Again recalling (10) we obtain that

$$\mathbb{P}(\Lambda_n^+) \geq \frac{K'}{\sqrt{2\pi}\sigma_{d,n}} \int_{-n}^{n} e^{-K''d^{c\lambda}} \exp(-(m_n - \lambda)^2/2\sigma_{d,n}^2) d\lambda.$$

The integrand here is infact a decreasing function of λ in the range $\lambda \in [\lambda', \lambda'+1]$, where λ' is from the first part of the proof. This gives a lower bound of

$$\frac{K'}{\sqrt{2\pi}\sigma_{d,n}} e^{-K''d^c d^{c\lambda'}} \exp(-(m_n - \lambda' - 1)^2/2\sigma_{d,n}^2).$$

So, we obtain the required lower bound in (2). □

5 Expected value of a typical vertex under positivity

Proof. of theorem 1.2. We want to compute $\mathbb{E}\left(\frac{S_n}{d^n} \mid \Lambda_n^+\right)$. Due to Lemma 3.2, this is equivalent to computing $\mathbb{E}\left(\frac{\sum_{v=1}^{d^n} \xi_v^n}{d^n} \mid \xi_u^n \geq 0 \ \forall\ v \in T_n\right) = \mathbb{E}\left(X \mid \max_{v\in T_n} \tilde{\phi}_v^n \leq X\right)$.

Upper Bound: We first split the expectation into two parts, one concerning the contribution of the right tail in the integral and the rest. We aim to show that the contribution of the right tail is negligible, thereby implying that the main contribution is from the rest, which gives an upper bound on the expectation. The tail here is motivated by the maximizer in Proposition 1.1.

$$
\begin{aligned}
\mathbb{E}\left(X \mid \max_{v\in T_n}\tilde\phi^n_v \le X\right) &= \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{m_n - b\log n} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&+ \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{m_n - b\log n}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx
\end{aligned}
$$

Let us call the first term as J_1 and the next one as J_2. We first want to show that the contribution of J_2 in the conditional expectation is negligible. We use a trivial upper bound on the tail probability in the numerator. Then we compute the integral which is the tail expectation of a normal.

$$
\begin{aligned}
J_2 &\le \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{m_n - b\log n}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}\frac{1}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}\int_{m_n - b\log n}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}dx \\
&= \frac{\sigma_{d,n}e^{-(m_n - b\log n)^2/2\sigma^2_{d,n}}}{\sqrt{2\pi}\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}
\end{aligned}
$$

So we end up showing that contribution from the right tail is negligible. We now move on to the rest part and obtain an upper bound for it. We use a general upper bound on x from the range of the integral, which we can do since the integral exists and is finite by the fact that absolute expectation of a normal exists.

$$
\begin{aligned}
J_1 &\le \frac{m_n - b\log n}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{m_n - b\log n} e^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&\le \frac{m_n - b\log n}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&= m_n - b\log n
\end{aligned}
$$

From (2) it is clear that on choosing b such that $b\log n \le \lambda'$ then the upper bound on the conditional expectation is $m_n - b\log n$.

Lower Bound: We apply a similar technique as in case of the upper bound, the only difference being that we look at the left tail instead, motivated by the left tail of the maximum of the Gaussian process.

$$
\begin{aligned}
\mathbb{E}\left(X \mid \max_{v\in T_n}\tilde\phi^n_v \le X\right) &= \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{m_n - \frac{3}{c}\log_d n} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx \\
&+ \frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{m_n - \frac{3}{c}\log_d n}^{\infty} xe^{-x^2/2\sigma^2_{d,n}}\frac{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le x)}{\mathbb{P}(\max_{v\in T_n}\tilde\phi^n_v \le X)}dx
\end{aligned}
$$

Let us call the first term as I_1 and the second as I_2.

When $x \in (-\infty, m_n - \frac{3}{c}\log_d n]$ then $\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x) \leq C' \exp(-C'' n^3)$ following (10). Also we have a lower bound on the probability of positivity, which gives the following bounds on I_1 and I_2.

$$| I_1 | \lesssim e^{\frac{1}{2\sigma_{d,n}^2}(m_n-\lambda')^2 + d^{c\lambda'}(\log \lambda' - \log \bar{p}/K) - C'' n^3} \int_{-\infty}^{\infty} | x | \, e^{-x^2/2\sigma_{d,n}^2} dx$$

where we ignore the constants. This shows that this term is negligible. Further,

$$\begin{aligned} I_2 &\geq \left(m_n - \frac{3}{c}\log_d n\right) \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n - \frac{3}{c}\log_d n}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx \\ &= \left(m_n - \frac{3}{c}\log_d n\right) \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx - o(1) \\ &= m_n - \frac{3}{c}\log_d n \end{aligned}$$

□

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